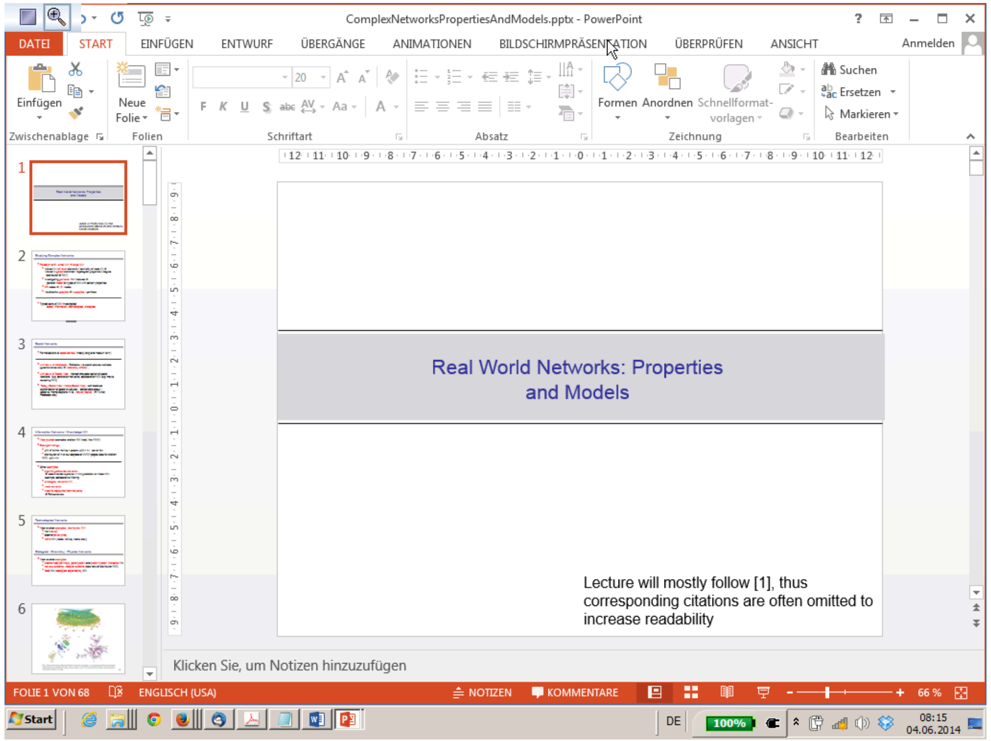
Script **generated by TTT**

Title: groh: profile1 (04.06.2014)

Date: Wed Jun 04 08:15:53 CEST 2014

Duration: 91:03 min

Pages: 83



Random Graph Models: Poisson Graph

- $G_{n,p}$: space of graphs with n nodes and each of the ½ n(n-1) edges appears with probability p
- p_k: probability that a node has degree k:

$$p_k = \binom{n}{k} p^k (1-p)^{n-k} \simeq \frac{z^k e^{-z}}{k!}$$

for n → ∞ and holding the mean degree of a node z=p(n-1) fixed
(Poisson approximation of Binomial distribution)
→ „Poisson random graphs“

Random Graph Models: Poisson Graph

- $G_{n,p}$: space of graphs with n nodes and each of the ½ n(n-1) edges appears with probability p
- p_k: probability that a node has degree k:

$$p_k = \binom{n}{k} p^k (1-p)^{n-k} \simeq \frac{z^k e^{-z}}{k!}$$

for n → ∞ and holding the mean degree of a node z=p(n-1) fixed
(Poisson approximation of Binomial distribution)
→ „Poisson random graphs“

Random Graph Models: Poisson Graph

- $G_{n,p}$: space of graphs with n nodes and
  each of the ½ n(n-1) edges appears with probability p

- p_k: probability that a node has degree k:

$$p_k = \binom{n}{k} p^k (1-p)^{n-k} \simeq \frac{z^k \mathrm{e}^{-z}}{k!}$$

for n → ∞ and holding the mean degree of a node z=p(n-1) fixed
(Poisson approximation of Binomial distribution)
→ „Poisson random graphs"

Random Graph Models: Poisson Graph

- $G_{n,p}$: space of graphs with n nodes and
  each of the ½ n(n-1) edges appears with probability p

- p_k: probability that a node has degree k:

$$p_k = \binom{n}{k} p^k (1-p)^{n-k} \simeq \frac{z^k \mathrm{e}^{-z}}{k!}$$

for n → ∞ and holding the mean degree of a node z=p(n-1) fixed
(Poisson approximation of Binomial distribution)
→ „Poisson random graphs"

Random Graph Models: Poisson Graph

- Given: property Q („is connected", „has diameter xyz" etc.) of $G_{n,p}$:
  „$G_{n,p}$ has property Q with high probability": P(Q|n,p) → 1 iff n → ∞
  (adaptated from [2] (which, in turn, is adaptated from [3]))

- In such models $G_{n,p}$ phase transitions exist for properties Q:
  „threshold function" q(n) (with q(n) → ∞ if n → ∞) so that:

$$\lim_{n\to\infty} P(Q|n,p) = \begin{cases} 0 & \text{if } \lim_{n\to\infty} p(n)/q(n) = 0 \\ 1 & \text{if } \lim_{n\to\infty} p(n)/q(n) = \infty \end{cases}$$

(adaptated from [3])

Random Graph Models: Poisson Graph

- Given: property Q („is connected", „has diameter xyz" etc.) of $G_{n,p}$:
  „$G_{n,p}$ has property Q with high probability": P(Q|n,p) → 1 iff n → ∞
  (adaptated from [2] (which, in turn, is adaptated from [3]))

- In such models $G_{n,p}$ phase transitions exist for properties Q:
  „threshold function" q(n) (with q(n) → ∞ if n → ∞) so that:

$$\lim_{n\to\infty} P(Q|n,p) = \begin{cases} 0 & \text{if } \lim_{n\to\infty} p(n)/q(n) = 0 \\ 1 & \text{if } \lim_{n\to\infty} p(n)/q(n) = \infty \end{cases}$$

(adaptated from [3])

Random Graph Models: Poisson Graph

- Given: property Q („is connected“, „has diameter xyz“ etc.) of $G_{n,p}$:
  „$G_{n,p}$ has property Q with high probability“: P(Q|n,p) → 1 iff n → ∞
  (adaptated from [2] (which, in turn, is adaptated from [3]))

- In such models $G_{n,p}$ phase transitions exist for properties Q:
  „threshold function“ q(n) (with q(n) → ∞ if n → ∞) so that:

$$\lim_{n\to\infty} P(Q|n,p) = \begin{cases} 0 & \text{if } \lim_{n\to\infty} p(n) / q(n) = 0 \\ 1 & \text{if } \lim_{n\to\infty} p(n) / q(n) = \infty \end{cases}$$

(adaptated from [3])

Random Graph Models: Poisson Graph

- Given: property Q („is connected“, „has diameter xyz“ etc.) of $G_{n,p}$:
  „$G_{n,p}$ has property Q with high probability“: P(Q|n,p) → 1 iff n → ∞
  (adaptated from [2] (which, in turn, is adaptated from [3]))

- In such models $G_{n,p}$ phase transitions exist for properties Q:
  „threshold function“ q(n) (with q(n) → ∞ if n → ∞) so that:

$$\lim_{n\to\infty} P(Q|n,p) = \begin{cases} 0 & \text{if } \lim_{n\to\infty} p(n) / q(n) = 0 \\ 1 & \text{if } \lim_{n\to\infty} p(n) / q(n) = \infty \end{cases}$$

(adaptated from [3])

Random Graph Models: Poisson Graph

Example: giant component / connectedness of $G_{n,p}$

- Let u be the fraction of nodes that do not belong to giant component X
  == probability for a given node i to be not in X

- probability for a given node i (with assumed degree k) to be not in X
  == probability that none of its neighbors is in X
  == u^k

- → u (k fixed) == u^k → $u = \sum_{k=0}^{\infty} p_k u^k = \mathrm{e}^{-z} \sum_{k=0}^{\infty} \frac{(zu)^k}{k!} = \mathrm{e}^{z(u-1)}$

- → fraction S of graph occupied by X is $S = 1 - u$ →

$$S = 1 - \mathrm{e}^{-zS}$$

Random Graph Models: Poisson Graph

Example: giant component / connectedness of $G_{n,p}$

- Let u be the fraction of nodes that do not belong to giant component X
  == probability for a given node i to be not in X

- probability for a given node i (with assumed degree k) to be not in X
  == probability that none of its neighbors is in X
  == u^k

- → u (k fixed) == u^k → $u = \sum_{k=0}^{\infty} p_k u^k = \mathrm{e}^{-z} \sum_{k=0}^{\infty} \frac{(zu)^k}{k!} = \mathrm{e}^{z(u-1)}$

- → fraction S of graph occupied by X is $S = 1 - u$ →

$$S = 1 - \mathrm{e}^{-zS}$$

Random Graph Models: Poisson Graph

Example: giant component / connectedness of $G_{n,p}$

- Let u be the fraction of nodes that do not belong to giant component X
  == probability for a given node i to be not in X
- probability for a given node i (with assumed degree k) to be not in X
  == probability that none of its neighbors is in X
  == u^k
- → u (k fixed) == u^k → $u = \sum_{k=0}^{\infty} p_k u^k = e^{-z} \sum_{k=0}^{\infty} \frac{(zu)^k}{k!} = e^{z(u-1)}$
- → fraction S of graph occupied by X is $S = 1 - u$ →

$$S = 1 - e^{-zS}$$

Random Graph Models: Poisson Graph

Example: giant component / connectedness of $G_{n,p}$

- Let u be the fraction of nodes that do not belong to giant component X
  == probability for a given node i to be not in X
- probability for a given node i (with assumed degree k) to be not in X
  == probability that none of its neighbors is in X
  == u^k
- → u (k fixed) == u^k → $u = \sum_{k=0}^{\infty} p_k u^k = e^{-z} \sum_{k=0}^{\infty} \frac{(zu)^k}{k!} = e^{z(u-1)}$
- → fraction S of graph occupied by X is $S = 1 - u$ →

$$S = 1 - e^{-zS}$$

Random Graph Models: Poisson Graph

Example: giant component / connectedness of $G_{n,p}$

- Let u be the fraction of nodes that do not belong to giant component X
  == probability for a given node i to be not in X
- probability for a given node i (with assumed degree k) to be not in X
  == probability that none of its neighbors is in X
  == u^k
- → u (k fixed) == u^k → $u = \sum_{k=0}^{\infty} p_k u^k = e^{-z} \sum_{k=0}^{\infty} \frac{(zu)^k}{k!} = e^{z(u-1)}$
- → fraction S of graph occupied by X is $S = 1 - u$ →

$$S = 1 - e^{-zS}$$

Random Graph Models: Poisson Graph

Example: giant component / connectedness of $G_{n,p}$

- Let u be the fraction of nodes that do not belong to giant component X
  == probability for a given node i to be not in X
- probability for a given node i (with assumed degree k) to be not in X
  == probability that none of its neighbors is in X
  == u^k
- → u (k fixed) == u^k → $u = \sum_{k=0}^{\infty} p_k u^k = e^{-z} \sum_{k=0}^{\infty} \frac{(zu)^k}{k!} = e^{z(u-1)}$
- → fraction S of graph occupied by X is $S = 1 - u$ →

$$S = 1 - e^{-zS}$$

Random Graph Models: Poisson Graph

Example: giant component / connectedness of $G_{n,p}$

- Let u be the fraction of nodes that do not belong to giant component X
  == probability for a given node i to be not in X
- probability for a given node i (with assumed degree k) to be not in X
  == probability that none of its neighbors is in X
  == u^k
- → u (k fixed) == u^k → $u = \sum_{k=0}^{\infty} p_k u^k = \mathrm{e}^{-z} \sum_{k=0}^{\infty} \frac{(zu)^k}{k!} = \mathrm{e}^{z(u-1)}$
- → fraction S of graph occupied by X is $S = 1 - u$ →

$$S = 1 - \mathrm{e}^{-zS}$$

Random Graph Models: Poisson Graph

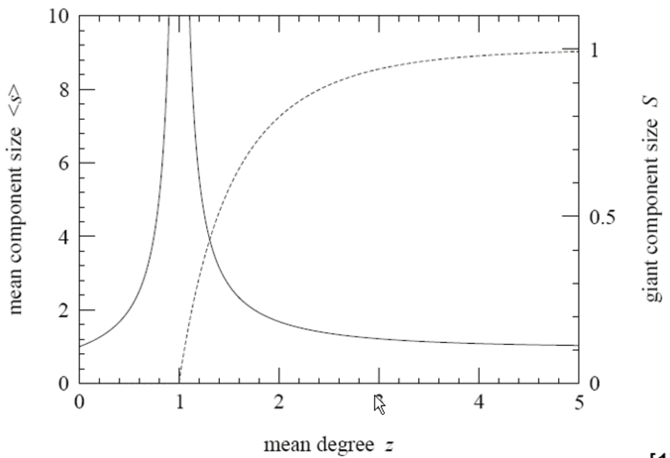
Example: giant component / connectedness of $G_{n,p}$

- Let u be the fraction of nodes that do not belong to giant component X
  == probability for a given node i to be not in X
- probability for a given node i (with assumed degree k) to be not in X
  == probability that none of its neighbors is in X
  == u^k
- → u (k fixed) == u^k → $u = \sum_{k=0}^{\infty} p_k u^k = \mathrm{e}^{-z} \sum_{k=0}^{\infty} \frac{(zu)^k}{k!} = \mathrm{e}^{z(u-1)}$
- → fraction S of graph occupied by X is $S = 1 - u$ →

$$S = 1 - \mathrm{e}^{-zS}$$

Random Graph Models: Poisson Graph

- $S = 1 - \mathrm{e}^{-zS}$
- mean size <s> of smaller rest components (no proof): $\langle s \rangle = \frac{1}{1 - z + zS}$

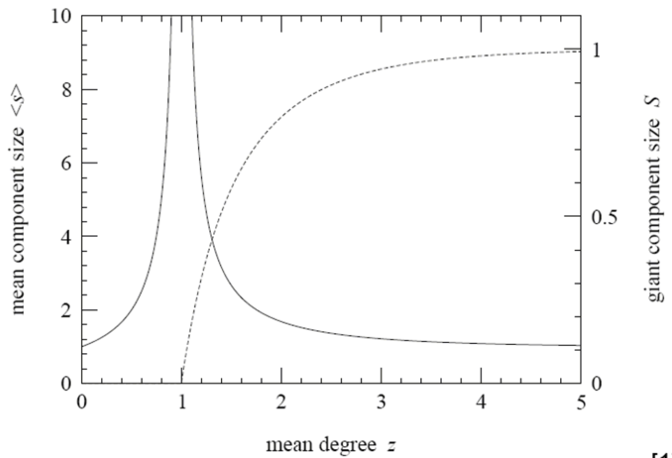


[1]

→ if the av degree z is larger than 1 (== if p ~ (1+ε)/n): X exists

Random Graph Models: Poisson Graph

- $S = 1 - \mathrm{e}^{-zS}$
- mean size <s> of smaller rest components (no proof): $\langle s \rangle = \frac{1}{1 - z + zS}$

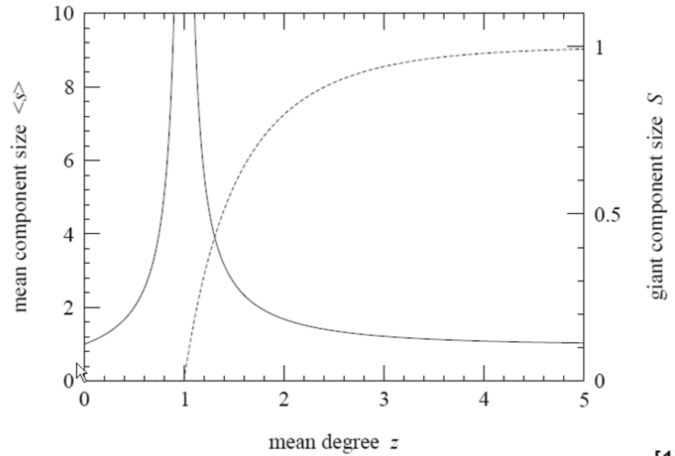


[1]

→ if the av degree z is larger than 1 (== if p ~ (1+ε)/n): X exists

Random Graph Models: Poisson Graph

- $S = 1 - e^{-zS}$
- mean size <s> of smaller rest components (no proof): $\langle s \rangle = \dfrac{1}{1 - z + zS}$

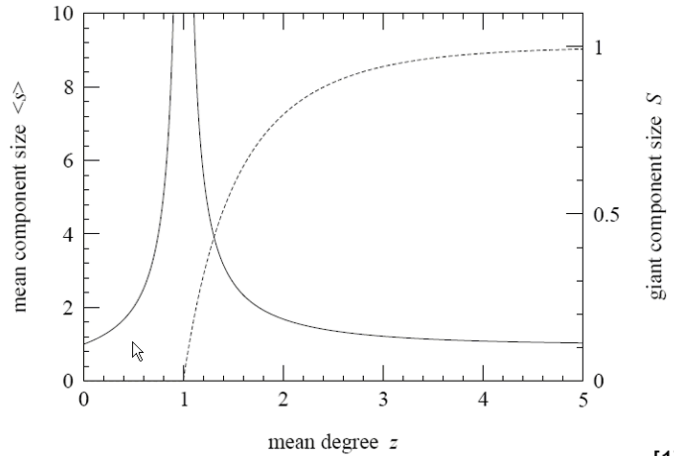


[1]

→ if the av degree z is larger than 1 (== if p ~ (1+ε)/n): X exists

Random Graph Models: Poisson Graph

- $S = 1 - e^{-zS}$
- mean size <s> of smaller rest components (no proof): $\langle s \rangle = \dfrac{1}{1 - z + zS}$

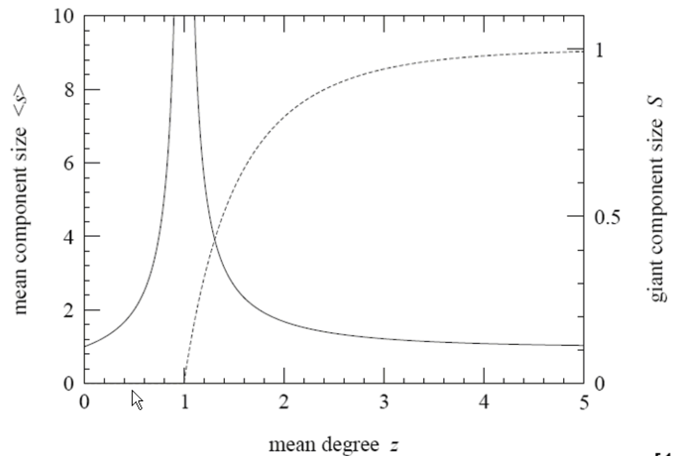


[1]

→ if the av degree z is larger than 1 (== if p ~ (1+ε)/n): X exists

Random Graph Models: Poisson Graph

- $S = 1 - e^{-zS}$
- mean size <s> of smaller rest components (no proof): $\langle s \rangle = \dfrac{1}{1 - z + zS}$

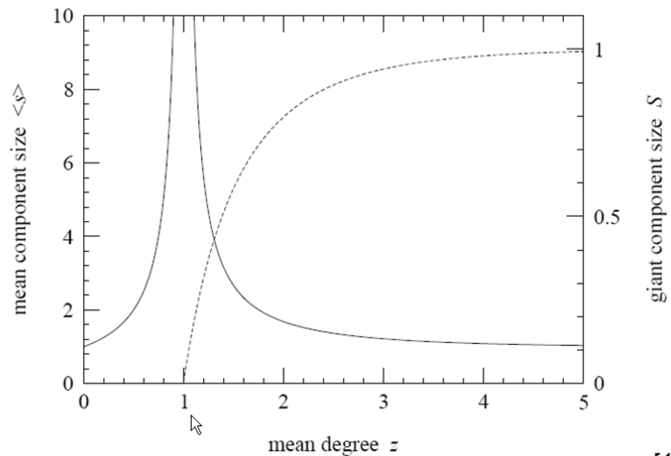


[1]

→ if the av degree z is larger than 1 (== if p ~ (1+ε)/n): X exists

Random Graph Models: Poisson Graph

- $S = 1 - e^{-zS}$
- mean size <s> of smaller rest components (no proof): $\langle s \rangle = \dfrac{1}{1 - z + zS}$

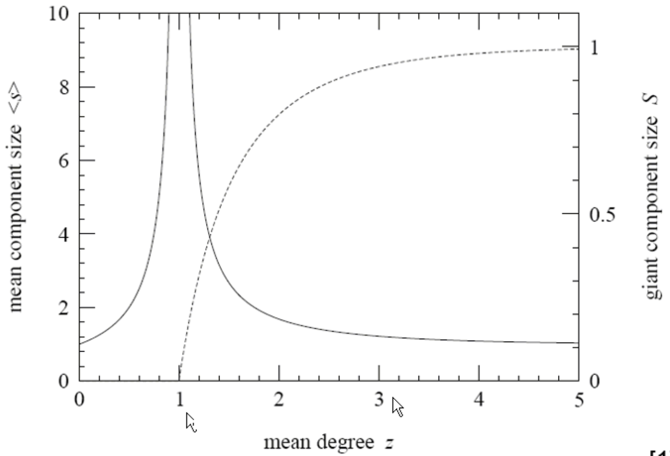


[1]

→ if the av degree z is larger than 1 (== if p ~ (1+ε)/n): X exists

Random Graph Models: Poisson Graph

- $S = 1 - e^{-zS}$
- mean size <s> of smaller rest components (no proof): $\langle s \rangle = \dfrac{1}{1 - z + zS}$

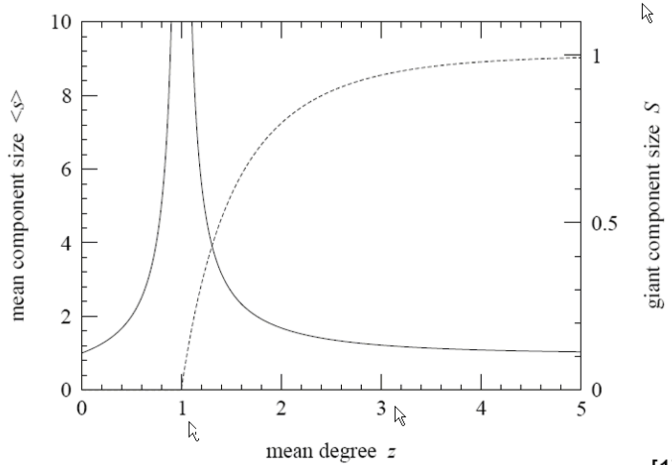


[1]

→ if the av degree z is larger than 1 (== if p ~ (1+ε)/n): X exists

Random Graph Models: Poisson Graph

- $S = 1 - e^{-zS}$
- mean size <s> of smaller rest components (no proof): $\langle s \rangle = \dfrac{1}{1 - z + zS}$

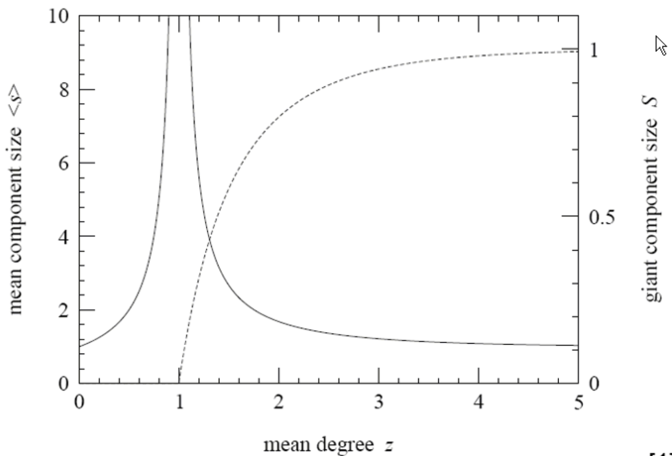


[1]

→ if the av degree z is larger than 1 (== if p ~ (1+ε)/n): X exists

Random Graph Models: Poisson Graph

- $S = 1 - e^{-zS}$
- mean size <s> of smaller rest components (no proof): $\langle s \rangle = \dfrac{1}{1 - z + zS}$



[1]

→ if the av degree z is larger than 1 (== if p ~ (1+ε)/n): X exists

Random Graph Models: Poisson Graph

Very coarse (!!!) estimation of diameter l of $G_{n,p}$:

- average degree of nodes: z
  → in a distance of d from a node i should be approximately z^d many nodes
  → if $z^d = n$: $d = l$
  → $l \sim \log n / \log z \sim \log n$ (if z is kept constant)

- For a more exact derivation of the result see references in [1]

- We see: it is not difficult (in terms of how large must connectivity be) to achieve small diameters

Random Graph Models: Poisson Graph

Very coarse (!!!) estimation of diameter l of $G_{n,p}$:

- average degree of nodes: z
  → in a distance of d from a node i should be approximately z^d many nodes
  → if $z^d = n$: $d = l$
  → $l \sim \log n / \log z \sim \log n$ (if z is kept constant)
- For a more exact derivation of the result see references in [1]
- We see: it is not difficult (in terms of how large must connectivity be) to achieve small diameters

Random Graph Models: Poisson Graph

Very coarse (!!!) estimation of diameter l of $G_{n,p}$:

- average degree of nodes: z
  → in a distance of d from a node i should be approximately z^d many nodes
  → if $z^d = n$: $d = l$
  → $l \sim \log n / \log z \sim \log n$ (if z is kept constant)
- For a more exact derivation of the result see references in [1]
- We see: it is not difficult (in terms of how large must connectivity be) to achieve small diameters

Random Graph Models: Poisson Graph

Unfortunately: small l is the only property in congruence with real world NW:

- Clustering coefficient $C^{(1)}$ of $G_{n,p}$:
  - Since $C^{(1)}$ is probability of transitivity and edges are "drawn" independently → $C^{(1)} = p = O(1/n)$ (if z is fixed, as usual)
  - C is usually much larger for real world NW:

	l (real)	l (random)	$C^{(2)}$ (real)	C (random)
Film collaboration	3.65	2.99	0.79	0.00027
Power Grid	18.7	12.4	0.08	0.005
C.elegans	2.65	2.25	0.28	0.05

[4]

- Degree distribution is Poisson and not power law

Random Graph Models: Poisson Graph

Unfortunately: small l is the only property in congruence with real world NW:

- Clustering coefficient $C^{(1)}$ of $G_{n,p}$:
  - Since $C^{(1)}$ is probability of transitivity and edges are "drawn" independently → $C^{(1)} = p = O(1/n)$ (if z is fixed, as usual)
  - C is usually much larger for real world NW:

	l (real)	l (random)	$C^{(2)}$ (real)	C (random)
Film collaboration	3.65	2.99	0.79	0.00027
Power Grid	18.7	12.4	0.08	0.005
C.elegans	2.65	2.25	0.28	0.05

[4]

- Degree distribution is Poisson and not power law

Random Graph Models: Poisson Graph

Unfortunately: small l is the only property in congruence with real world NW:

- Clustering coefficient $C^{(1)}$ of $G_{n,p}$:
  - Since $C^{(1)}$ is probability of transitivity and edges are "drawn" independently → $C^{(1)} = p = O(1/n)$ (if z is fixed, as usual)
  - C is usually much larger for real world NW:

	l (real)	l (random)	$C^{(2)}$ (real)	C (random)
Film collaboration	3.65	2.99	0.79	0.00027
Power Grid	18.7	12.4	0.08	0.005
C.elegans	2.65	2.25	0.28	0.05

[4]

- Degree distribution is Poisson and not power law

Random Graph Models: More Refined Models

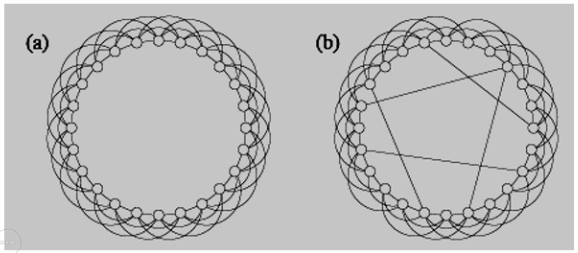
- Instead of having connection probability p as in Poisson $G_{n,p}$: demand certain degree distributions p_k (e.g. power law) → „configuration model"
- → results are promising but still not in full congruence with real world NW
- → still many difficult open problems
- still not accounted for: transitivity (high clustering coefficient)

Watts Strogatz Model

- Great problem of random graphs: high clustering coeff. / transitivity does not occur for simple models
- → Watts & Strogatz 1998: Small World Model
  - L nodes in regular D-dim. lattice + periodic boundary cond.; D=1: Ring
  - each node connected to neighbors in lattice at distance of most k
    → total number of edges = L k
  - „rewiring" of edges with probability p

before rewiring after rewiring

D = 1; k = 3



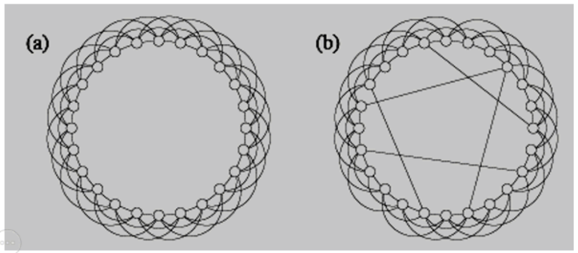
[1]

Watts Strogatz Model

- Great problem of random graphs: high clustering coeff. / transitivity does not occur for simple models
- → Watts & Strogatz 1998: Small World Model
  - L nodes in regular D-dim. lattice + periodic boundary cond.; D=1: Ring
  - each node connected to neighbors in lattice at distance of most k
    → total number of edges = L k
  - „rewiring" of edges with probability p

before rewiring after rewiring

D = 1; k = 3



[1]

Watts Strogatz Model

- Great problem of random graphs: high clustering coeff. / transitivity does not occur for simple models

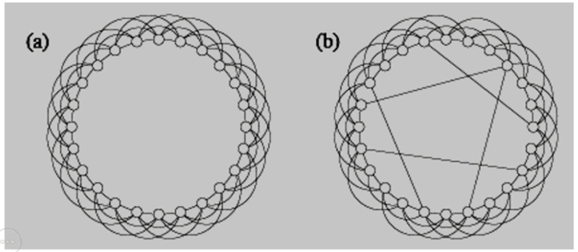
---

- → Watts & Strogatz 1998: Small World Model
  - L nodes in regular D-dim. lattice + periodic boundary cond.; D=1: Ring
  - each node connected to neighbors in lattice at distance of most k

    → total number of edges = L k
  - „rewiring“ of edges with probability p

before rewiring after rewiring

D = 1; k = 3



[1]

Watts Strogatz Model

- Great problem of random graphs: high clustering coeff. / transitivity does not occur for simple models

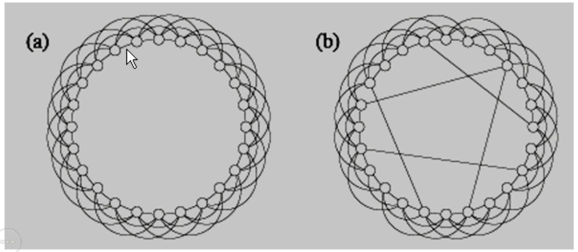
---

- → Watts & Strogatz 1998: Small World Model
  - L nodes in regular D-dim. lattice + periodic boundary cond.; D=1: Ring
  - each node connected to neighbors in lattice at distance of most k

    → total number of edges = L k
  - „rewiring“ of edges with probability p

before rewiring after rewiring

D = 1; k = 3



[1]

Watts Strogatz Model

- Great problem of random graphs: high clustering coeff. / transitivity does not occur for simple models

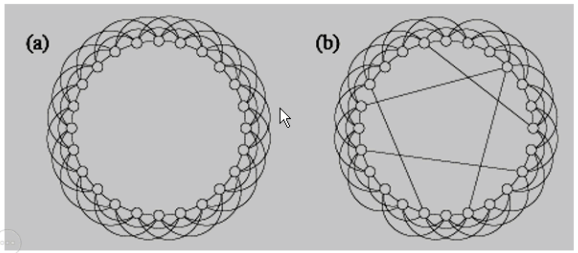
---

- → Watts & Strogatz 1998: Small World Model
  - L nodes in regular D-dim. lattice + periodic boundary cond.; D=1: Ring
  - each node connected to neighbors in lattice at distance of most k

    → total number of edges = L k
  - „rewiring“ of edges with probability p

before rewiring after rewiring

D = 1; k = 3



[1]

Watts Strogatz Model

- Great problem of random graphs: high clustering coeff. / transitivity does not occur for simple models

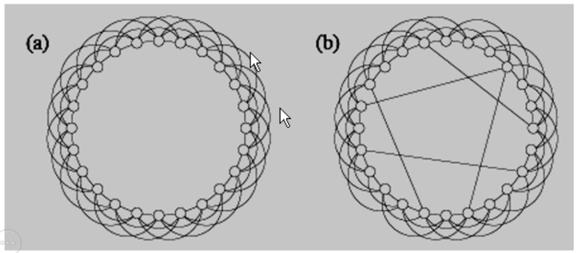
---

- → Watts & Strogatz 1998: Small World Model
  - L nodes in regular D-dim. lattice + periodic boundary cond.; D=1: Ring
  - each node connected to neighbors in lattice at distance of most k

    → total number of edges = L k
  - „rewiring“ of edges with probability p

before rewiring after rewiring

D = 1; k = 3



[1]

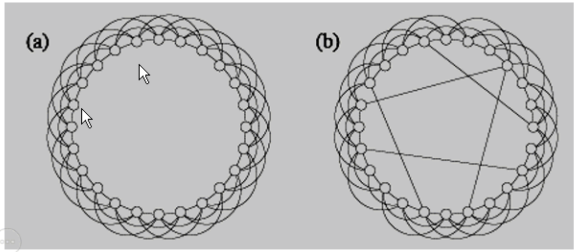
Watts Strogatz Model

- Great problem of random graphs: high clustering coeff. / transitivity does not occur for simple models

- → Watts & Strogatz 1998: Small World Model
  - L nodes in regular D-dim. lattice + periodic boundary cond.; D=1: Ring
  - each node connected to neighbors in lattice at distance of most k
    → total number of edges = L k
  - „rewiring“ of edges with probability p

before rewiring after rewiring

D = 1; k = 3



[1]

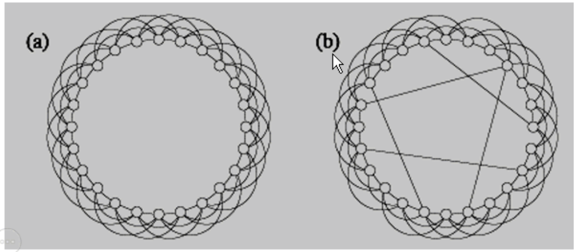
Watts Strogatz Model

- Great problem of random graphs: high clustering coeff. / transitivity does not occur for simple models

- → Watts & Strogatz 1998: Small World Model
  - L nodes in regular D-dim. lattice + periodic boundary cond.; D=1: Ring
  - each node connected to neighbors in lattice at distance of most k
    → total number of edges = L k
  - „rewiring“ of edges with probability p

before rewiring after rewiring

D = 1; k = 3



[1]

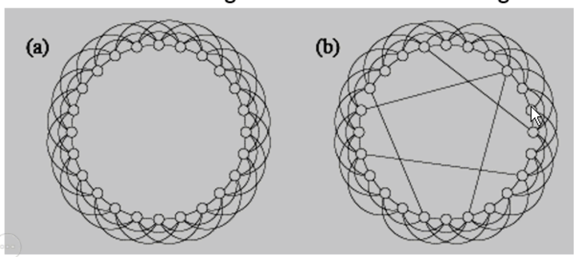
Watts Strogatz Model

- Great problem of random graphs: high clustering coeff. / transitivity does not occur for simple models

- → Watts & Strogatz 1998: Small World Model
  - L nodes in regular D-dim. lattice + periodic boundary cond.; D=1: Ring
  - each node connected to neighbors in lattice at distance of most k
    → total number of edges = L k
  - „rewiring“ of edges with probability p

before rewiring after rewiring

D = 1; k = 3



[1]

Watts Strogatz Model

p: transition between regular lattice and sth. like a random graph:
(for D=1:)

- p=0: regular lattice:
  - $C = C^{(1)} = (3k-3)/(4k-2)$ → ¾ for $k \to \infty$ → clustering coeff. „ok“
  - $l = L / 4k$ for $L \to \infty$ → no small world effect

- p=1: similar to a random graph:
  - $C \sim 2k / L$ for $L \to \infty$ → clustering coeff too small
  - $l = \log L / \log k$ for $L \to \infty$ → small world effect.

Watts Strogatz Model

p: transition between regular lattice and sth. like a random graph:
(for D=1:)

- p=0: regular lattice:
  - $C = C^{(1)} = (3k-3)/(4k-2)$ → ¾ for $k \to \infty$ → clustering coeff. „ok“
  - $l = L / 4k$ for $L \to \infty$ → no small world effect
- p=1: similar to a random graph:
  - $C \sim 2k / L$ for $L \to \infty$ → clustering coeff too small
  - $l = \log L / \log k$ for $L \to \infty$ → small world effect.

Watts Strogatz Model

p: transition between regular lattice and sth. like a random graph:
(for D=1:)

- p=0: regular lattice:
  - $C = C^{(1)} = (3k-3)/(4k-2)$ → ¾ for $k \to \infty$ → clustering coeff. „ok“
  - $l = L / 4k$ for $L \to \infty$ → no small world effect
- p=1: similar to a random graph:
  - $C \sim 2k / L$ for $L \to \infty$ → clustering coeff too small
  - $l = \log L / \log k$ for $L \to \infty$ → small world effect.

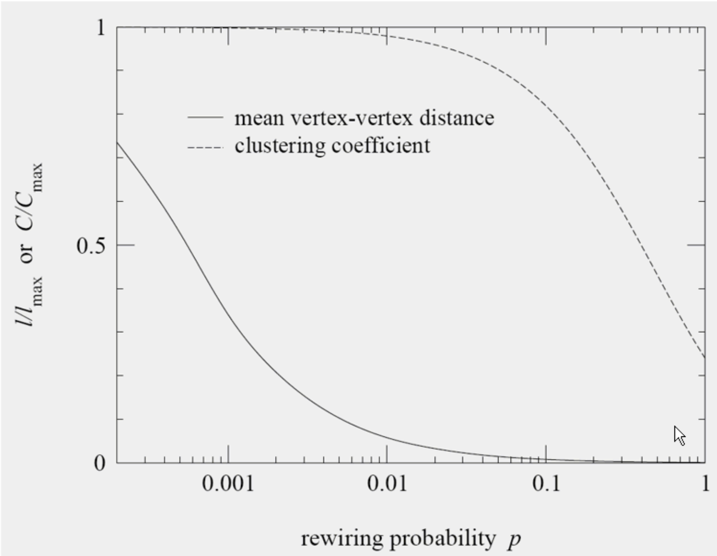
Watts Strogatz Model

p: transition between regular lattice and sth. like a random graph:
(for D=1:)

- p=0: regular lattice:
  - $C = C^{(1)} = (3k-3)/(4k-2)$ → ¾ for $k \to \infty$ → clustering coeff. „ok“
  - $l = L / 4k$ for $L \to \infty$ → no small world effect
- p=1: similar to a random graph:
  - $C \sim 2k / L$ for $L \to \infty$ → clustering coeff too small
  - $l = \log L / \log k$ for $L \to \infty$ → small world effect.

Watts Strogatz Model

- Interesting area: intermediate values for p:



[1]

Watts Strogatz Model

p: transition between regular lattice and sth. like a random graph:
(for D=1:)

- p=0: regular lattice:
  - $C = C^{(1)} = (3k-3)/(4k-2)$ → ¾ for $k \to \infty$ → clustering coeff. „ok"
  - $l = L / 4k$ for $L \to \infty$ → no small world effect
- p=1: similar to a random graph:
  - $C \sim 2k / L$ for $L \to \infty$ → clustering coeff too small
  - $l = \log L / \log k$ for $L \to \infty$ → small world effect.

Watts Strogatz Model

p: transition between regular lattice and sth. like a random graph:
(for D=1:)

- p=0: regular lattice:
  - $C = C^{(1)} = (3k-3)/(4k-2)$ → ¾ for $k \to \infty$ → clustering coeff. „ok"
  - $l = L / 4k$ for $L \to \infty$ → no small world effect
- p=1: similar to a random graph:
  - $C \sim 2k / L$ for $L \to \infty$ → clustering coeff too small
  - $l = \log L / \log k$ for $L \to \infty$ → small world effect.

Watts Strogatz Model

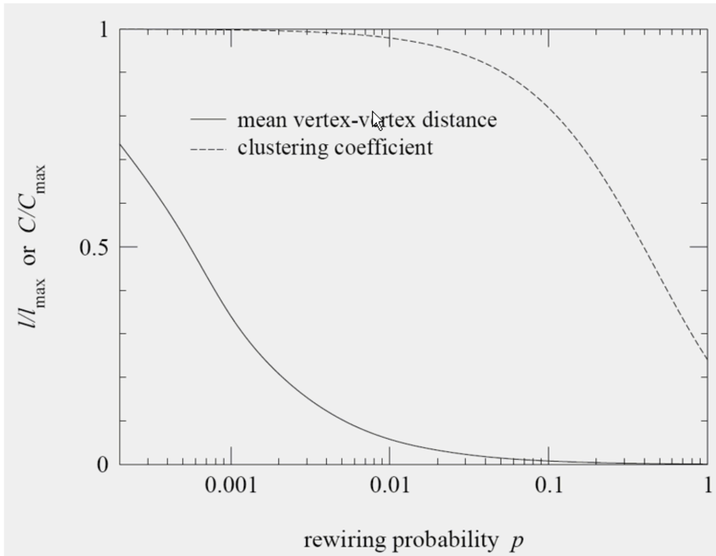
- Variants: -(1)- rewire both „ends" of edges + allow self-edges +....
  → math.easier

  -(2)- only add **additional** shortcut edges (no rewiring)

- For (2):
  - mean total number of shortcuts = L k p
  - mean degree of each node = 2k(1+p)

Watts Strogatz Model

- Interesting area: intermediate values for p:



[1]

Watts Strogatz Model

- Clustering coefficient:

$$C = \frac{3(k-1)}{2(2k-1)}(1-p)^3 \qquad (*)$$

$$C = \frac{3(k-1)}{2(2k-1)+4kp(p+2)} \qquad (**)$$

- almost constant for k→∞ and p≠1
- in good congruence with observed values for real world NW

(*): original model; (**) variant (2)

Watts Strogatz Model

- Clustering coefficient:

$$C = \frac{3(k-1)}{2(2k-1)}(1-p)^3 \qquad (*)$$

$$C = \frac{3(k-1)}{2(2k-1)+4kp(p+2)} \qquad (**)$$

- almost constant for k→∞ and p≠1
- in good congruence with observed values for real world NW

(*): original model; (**) variant (2)

Watts Strogatz Model

- Degree distribution for variant (2):

$$p_j = \binom{L}{j-2k}\left[\frac{2kp}{L}\right]^{j-2k}\left[1-\frac{2kp}{L}\right]^{L-j+2k} \qquad (**)$$

for $j \geq 2k$, and $p_j = 0$ for $j < 2k$.

- in variant (2): p defined so that :
  - -- mean number of added shortcuts == Lkp
  - -- and the mean degree == 2k + 2kp (2k from lattice plus 2kp added random shortcuts)
  - -- number of shortcuts is binomially distrib.
- Expectation of Binomial distribution: $E(X \sim B(L,\tilde{p})) = L\,\tilde{p}$

  → $\tilde{p} = \frac{2kp}{L}$

(*): original model; (**) variant (2)

Watts Strogatz Model

- Degree distribution for variant (2):

$$p_j = \binom{L}{j-2k}\left[\frac{2kp}{L}\right]^{j-2k}\left[1-\frac{2kp}{L}\right]^{L-j+2k} \qquad (**)$$

for $j \geq 2k$, and $p_j = 0$ for $j < 2k$.

- in variant (2): p defined so that :
  - -- mean number of added shortcuts == Lkp
  - -- and the mean degree == 2k + 2kp (2k from lattice plus 2kp added random shortcuts)
  - -- number of shortcuts is binomially distrib.
- Expectation of Binomial distribution: $E(X \sim B(L,\tilde{p})) = L\,\tilde{p}$

  → $\tilde{p} = \frac{2kp}{L}$

(*): original model; (**) variant (2)

Watts Strogatz Model

- Degree distribution for variant (2):

$$p_j = \binom{L}{j-2k}\left[\frac{2kp}{L}\right]^{j-2k}\left[1-\frac{2kp}{L}\right]^{L-j+2k} \qquad (**)$$

for $j \geq 2k$, and $p_j = 0$ for $j < 2k$.

- in variant (2): p defined so that :
  - -- mean number of added shortcuts == Lkp
  - -- and the mean degree == 2k + 2kp
    (2k from lattice plus 2kp added random shortcuts)
  - -- number of shortcuts is binomially distrib.

- Expectation of Binomial distribution: $E(X \sim B(L,\tilde{p})) = L\,\tilde{p}$

  $\rightarrow \tilde{p} = \frac{2kp}{L}$

(*): original model; (**) variant (2)

Watts Strogatz Model

- Degree distribution for variant (2):

$$p_j = \binom{L}{j-2k}\left[\frac{2kp}{L}\right]^{j-2k}\left[1-\frac{2kp}{L}\right]^{L-j+2k} \qquad (**)$$

for $j \geq 2k$, and $p_j = 0$ for $j < 2k$.

- in variant (2): p defined so that :
  - -- mean number of added shortcuts == Lkp
  - -- and the mean degree == 2k + 2kp
    (2k from lattice plus 2kp added random shortcuts)
  - -- number of shortcuts is binomially distrib.

- Expectation of Binomial distribution: $E(X \sim B(L,\tilde{p})) = L\,\tilde{p}$

  $\rightarrow \tilde{p} = \frac{2kp}{L}$

(*): original model; (**) variant (2)

Watts Strogatz Model

- Degree distribution for original model (without proof):

$$p_j = \sum_{n=0}^{\min(j-k,k)} \binom{k}{n}(1-p)^n p^{k-n}\frac{(pk)^{j-k-n}}{(j-k-n)!}\,e^{-pk} \qquad (*)$$

for $j \geq k$, and $p_j = 0$ for $j < k$.

(*): original model; (**) variant (2)

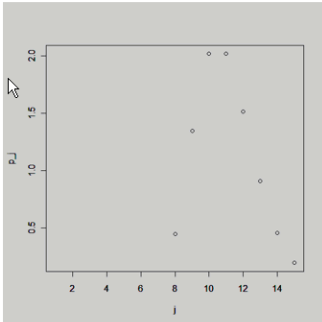
Watts Strogatz Model

- Degree distribution:

$$p_j = \binom{L}{j-2k}\left[\frac{2kp}{L}\right]^{j-2k}\left[1-\frac{2kp}{L}\right]^{L-j+2k} \qquad (**)$$

- Poisson approximation (justified):

$$p_j = \exp(-2kp)\frac{(2kp)^{j-2k}}{(j-2k)!}$$



k=3,p=0.5

- $\rightarrow$ almost constant
  $\rightarrow$ not in congruence with real world NW (power laws etc.)

(*): original model; (**) variant (2)

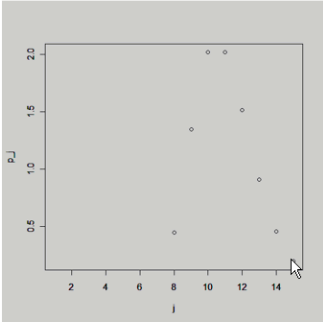
Watts Strogatz Model

- Degree distribution:

$$p_j = \binom{L}{j-2k}\left[\frac{2kp}{L}\right]^{j-2k}\left[1-\frac{2kp}{L}\right]^{L-j+2k} \qquad (**)$$

  - Poisson approximation (justified):

$$p_j = \exp(-2kp)\frac{(2kp)^{j-2k}}{(j-2k)!}$$



k=3,p=0.5

  - → almost constant
    → not in congruence with real world NW (power laws etc.)

(*): original model; (**) variant (2)

Watts Strogatz Model

- Average Path Length
  - Calculation is very hard, no precise results known as of 2010

- Approximation (1):

$$\ell = \xi g(L/\xi)$$

with $g(x) \sim \begin{cases} x & \text{for } x \gg 1 \\ \log x & \text{for } x \ll 1 \end{cases}$

and $\xi \sim p^{-\tau}$ for p→0

reproduces:

$\ell = L/4k$
(large world)
for p → 0 (regular lattice)

and

$\ell \sim \log L$
(small world)
for p → 1 (random)

Watts Strogatz Model

- Approximation (2) :

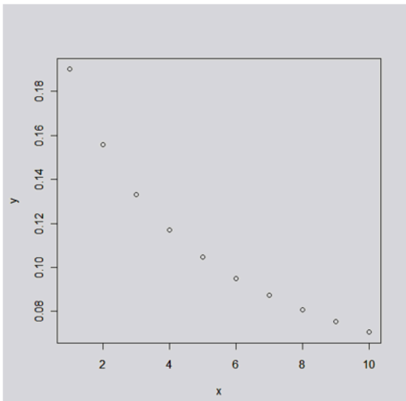
$$\ell = \frac{L}{k} f(\underbrace{Lkp}_{\text{mean number of shortcuts}})$$

== up to factor k same as Approx (1) for

$$\xi = 1/kp \text{ and } \bar{g}(x) = x f(x)$$

independent investigations yield approximation

$$f(x) = \frac{1}{2\sqrt{x^2+2x}}\tanh^{-1}\sqrt{\frac{x}{x+2}}$$



Watts Strogatz Model

- Approximation (2) :

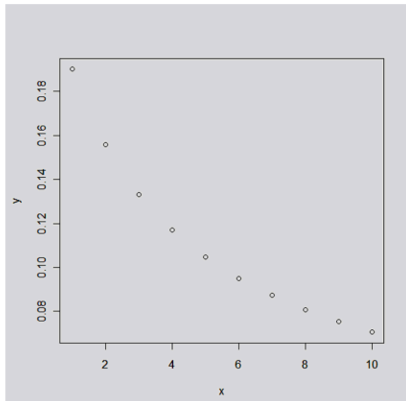
$$\ell = \frac{L}{k} f(\underbrace{Lkp}_{\text{mean number of shortcuts}})$$

== up to factor k same as Approx (1) for

$$\xi = 1/kp \text{ and } \bar{g}(x) = x f(x)$$

independent investigations yield approximation

$$f(x) = \frac{1}{2\sqrt{x^2+2x}}\tanh^{-1}\sqrt{\frac{x}{x+2}}$$



Price‘s Model

- Basic principle:
  „the rich get richer“
  „Matthew effect“ („For to every one that hath shall be given...“ Bible: Mt25:29)
  „preferential attachment“

- Assume directed citation NW:
  - p_k: fraction of nodes with in-degree k,
  - each node (paper) has av. out degree m
  - mean out-deg. $\stackrel{!}{=}$ mean in-deg. → $\sum_k k p_k = m$

- iteratively build graph by adding new vertices (and associated directed (out)edges from these nodes)

Price‘s Model

- probability for a paper X to get cited by a new paper is proportional to number of existing citations of X (X‘s in-degree)

  - initial „starting in-degree“ k_0=1
  - → prob. that new edge attaches to any node with in-deg. k ==

$$\frac{(k+1)p_k}{\sum_k (k+1)p_k} = \frac{(k+1)p_k}{m+1}$$

  - Since mean number of out-edges per added vertex == m → mean number of new in-edges to a node with current in-degree k is ==

$$x = \frac{(k+1)p_k}{m+1}\, m$$

  - mean number of nodes with in-degree k (which is np_k) decreases by x because their in-degree changes to k+1

Price‘s Model

- probability for a paper X to get cited by a new paper is proportional to number of existing citations of X (X‘s in-degree)

  - initial „starting in-degree“ k_0=1
  - → prob. that new edge attaches to any node with in-deg. k ==

$$\frac{(k+1)p_k}{\sum_k (k+1)p_k} = \frac{(k+1)p_k}{m+1}$$

  - Since mean number of out-edges per added vertex == m → mean number of new in-edges to a node with current in-degree k is ==

$$x = \frac{(k+1)p_k}{m+1}\, m$$

  - mean number of nodes with in-degree k (which is np_k) decreases by x because their in-degree changes to k+1

Price‘s Model

- Basic principle:
  „the rich get richer“
  „Matthew effect“ („For to every one that hath shall be given...“ Bible: Mt25:29)
  „preferential attachment“

- Assume directed citation NW:
  - p_k: fraction of nodes with in-degree k,
  - each node (paper) has av. out degree m
  - mean out-deg. $\stackrel{!}{=}$ mean in-deg. → $\sum_k k p_k = m$

- iteratively build graph by adding new vertices (and associated directed (out)edges from these nodes)

Price's Model

- probability for a paper X to get cited by a new paper is proportional to number of existing citations of X (X's in-degree)

  - initial „starting in-degree" k_0=1
  - → prob. that new edge attaches to any node with in-deg. k ==

$$\frac{(k+1)p_k}{\sum_k (k+1)p_k} = \frac{(k+1)p_k}{m+1}$$

  - Since mean number of out-edges per added vertex == m → mean number of new in-edges to a node with current in-degree k is ==

$$x = \frac{(k+1)p_k}{m+1} m$$

  - mean number of nodes with in-degree k (which is np_k) decreases by x because their in-degree changes to k+1

Price's Model

- probability for a paper X to get cited by a new paper is proportional to number of existing citations of X (X's in-degree)

  - initial „starting in-degree" k_0=1
  - → prob. that new edge attaches to any node with in-deg. k ==

$$\frac{(k+1)p_k}{\sum_k (k+1)p_k} = \frac{(k+1)p_k}{m+1}$$

  - Since mean number of out-edges per added vertex == m → mean number of new in-edges to a node with current in-degree k is ==

$$x = \frac{(k+1)p_k}{m+1} m$$

  - mean number of nodes with in-degree k (which is np_k) decreases by x because their in-degree changes to k+1

Price's Model

- probability for a paper X to get cited by a new paper is proportional to number of existing citations of X (X's in-degree)

  - initial „starting in-degree" k_0=1
  - → prob. that new edge attaches to any node with in-deg. k ==

$$\frac{(k+1)p_k}{\sum_k (k+1)p_k} = \frac{(k+1)p_k}{m+1}$$

  - Since mean number of out-edges per added vertex == m → mean number of new in-edges to a node with current in-degree k is ==

$$x = \frac{(k+1)p_k}{m+1} m$$

  - mean number of nodes with in-degree k (which is np_k) decreases by x because their in-degree changes to k+1

Price's Model

from previous slide:

- mean number of nodes with in-degree k (which is np_k) decreases by x because their in-degree changes to k+1

- mean number of nodes with in-degree k also increases because of nodes having previously k-1 and now have k

- → the net change in the quantity np_k per added vertex satisfies:

$$(n+1)p_{k,n+1} - np_{k,n} = \left[kp_{k-1,n} - (k+1)p_{k,n}\right]\frac{m}{m+1}$$

for $k \geq 1$, or

$$(n+1)p_{0,n+1} - np_{0,n} = 1 - p_{0,n}\frac{m}{m+1},$$

for $k = 0$.

Price's Model

from previous slide:

- mean number of nodes with in-degree k (which is np_k) decreases by x because their in-degree changes to k+1
- mean number of nodes with in-degree k also increases because of nodes having previously k-1 and now have k
- → the net change in the quantity np_k per added vertex satisfies:

$$(n+1)p_{k,n+1} - np_{k,n} = \left[kp_{k-1,n} - (k+1)p_{k,n}\right]\frac{m}{m+1}$$

for $k \geq 1$, or

$$(n+1)p_{0,n+1} - np_{0,n} = 1 - p_{0,n}\frac{m}{m+1},$$

for $k = 0$.

Price's Model

- probability for a paper X to get cited by a new paper is proportional to number of existing citations of X (X's in-degree)
  - initial „starting in-degree" k_0=1
  - → prob. that new edge attaches to any node with in-deg. k ==

$$\frac{(k+1)p_k}{\sum_k (k+1)p_k} = \frac{(k+1)p_k}{m+1}$$

  - Since mean number of out-edges per added vertex == m → mean number of new in-edges to a node with current in-degree k is ==

$$x = \frac{(k+1)p_k}{m+1}m$$

  - mean number of nodes with in-degree k (which is np_k) decreases by x because their in-degree changes to k+1

Price's Model

from previous slide:

- mean number of nodes with in-degree k (which is np_k) decreases by x because their in-degree changes to k+1
- mean number of nodes with in-degree k also increases because of nodes having previously k-1 and now have k
- → the net change in the quantity np_k per added vertex satisfies:

$$(n+1)p_{k,n+1} - np_{k,n} = \left[kp_{k-1,n} - (k+1)p_{k,n}\right]\frac{m}{m+1}$$

for $k \geq 1$, or

$$(n+1)p_{0,n+1} - np_{0,n} = 1 - p_{0,n}\frac{m}{m+1},$$

for $k = 0$.

Price's Model

- Computing stationary solutions $p_{k,n+1} = p_{k,n} = p_k$ of this equation we find:

$$p_k \sim k^{-(2+1/m)} \qquad \text{for } n \to \infty$$

- → the desired power law distribution
- we see: „the rich get richer" → power law

Price's Model

- Computing stationary solutions $p_{k,n+1} = p_{k,n} = p_k$ of this equation we find:

$$p_k \sim k^{-(2+1/m)} \qquad \text{for } n \rightarrow \infty$$

- → the desired power law distribution
- we see: „the rich get richer" → power law

Price's Model

- Computing stationary solutions $p_{k,n+1} = p_{k,n} = p_k$ of this equation we find:

$$p_k \sim k^{-(2+1/m)} \qquad \text{for } n \rightarrow \infty$$

- → the desired power law distribution
- we see: „the rich get richer" → power law

Barabasi-Albert Model and Price's Model

- crucial: linear preferential attachment
- found in a number of real world NW (e.g. citation NW)
- Barabasi-Albert: undirected (not like WWW)
- directed Barabasi Albert: attachment prop to sum of out and in-degree: not realistic for e.g. the WWW but for social NW?!
- Price: generates directed acyclic graph: not realistic for SN and WWW
- out-degree of WWW: power-law, Price + BA: constant

Barabasi-Albert Model and Price's Model

- crucial: linear preferential attachment
- found in a number of real world NW (e.g. citation NW)
- Barabasi-Albert: undirected (not like WWW)
- directed Barabasi Albert: attachment prop to sum of out and in-degree: not realistic for e.g. the WWW but for social NW?!
- Price: generates directed acyclic graph: not realistic for SN and WWW
- out-degree of WWW: power-law, Price + BA: constant

Barabasi-Albert Model and Price‘s Model

- crucial: linear preferential attachment
- found in a number of real world NW (e.g. citation NW)
- Barabasi-Albert: undirected (not like WWW)
- directed Barabasi Albert: attachment prop to sum of out and in-degree: not realistic for e.g. the WWW but for social NW?!
- Price: generates directed acyclic graph: not realistic for SN and WWW
- out-degree of WWW: power-law, Price + BA: constant

Barabasi-Albert Model and Price‘s Model

- crucial: linear preferential attachment
- found in a number of real world NW (e.g. citation NW)
- Barabasi-Albert: undirected (not like WWW)
- directed Barabasi Albert: attachment prop to sum of out and in-degree: not realistic for e.g. the WWW but for social NW?!
- Price: generates directed acyclic graph: not realistic for SN and WWW
- out-degree of WWW: power-law, Price + BA: constant

Barabasi-Albert Model and Price‘s Model

- crucial: linear preferential attachment
- found in a number of real world NW (e.g. citation NW)
- Barabasi-Albert: undirected (not like WWW)
- directed Barabasi Albert: attachment prop to sum of out and in-degree: not realistic for e.g. the WWW but for social NW?!
- Price: generates directed acyclic graph: not realistic for SN and WWW
- out-degree of WWW: power-law, Price + BA: constant

Barabasi-Albert Model and Price‘s Model

- crucial: linear preferential attachment
- found in a number of real world NW (e.g. citation NW)
- Barabasi-Albert: undirected (not like WWW)
- directed Barabasi Albert: attachment prop to sum of out and in-degree: not realistic for e.g. the WWW but for social NW?!
- Price: generates directed acyclic graph: not realistic for SN and WWW
- out-degree of WWW: power-law, Price + BA: constant