Script generated by TTT

groh: profile1 (30.06.2015) Title:

Tue Jun 30 15:04:44 CEST 2015 Date:

Duration: 73:02 min

Pages: 83

es in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

- Another example: Battle of the sexes
- Man & Woman: Ballet or Football

	В	F
F	0, 0	2, 1
В	1, 2	0, 0

- Another example: Game of chicken
- Driver 1 & Driver 2: Tough or Weak

	T	W
Т	-1,-1	2, 1
W	1, 2	0, 0

Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

- Another example: Battle of the sexes
- Two pure NE: (F;F) and (B;B)
- One mixed NE: Indifference condition
- \rightarrow Let $\sigma_1(F)=x$ and $\sigma_2(B)=y$

Player 1's indifference:

 $0 y + 2(1-y) = 1 y + 0 (1-y) \rightarrow y=2/3$

Player 2's indifference:

 $0 \times + 2(1-x) = 1 \times + 0 (1-x) \rightarrow x=2/3$

→ Mixed NE: ((2/3, 1/3); (2/3, 1/3))

- Another example: Game of chicken
- (same reasoning) →

Mixed NE: ((1/2, 1/2); (1/2, 1/2))

	В	F
F	0, 0	2, 1
В	1, 2	0, 0

	Т	W
Т	-1,-1	2, 1
W	1, 2	0, 0

es in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

- Another example: Battle of the sexes
- Two pure NE: (F;F) and (B;B)
- One mixed NE: Indifference condition \rightarrow Let $\sigma_1(F)=x$ and $\sigma_2(B)=y$ Player 1's indifference:

 $0 y + 2(1-y) = 1 y + 0 (1-y) \rightarrow y=2/3$

Player 2's indifference:

 $0 \times + 2(1-x) = 1 \times + 0 (1-x) \rightarrow x=2/3$ \rightarrow Mixed NE: ((2/3, 1/3); (2/3, 1/3))

- Another example: Game of chicken
- (same reasoning) →

Mixed NE: ((1/2, 1/2); (1/2, 1/2))

	В	F
Ę	0, 0	2, 1
В	1, 2	0, 0

	Т	W
Т	-1,-1	2, 1
W	1, 2	0, 0



- Sames in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

- Another example: Battle of the sexes
- Two pure NE: (F:F) and (B:B)
- One mixed NE: Indifference condition
- \rightarrow Let $\sigma_1(F)=x$ and $\sigma_2(B)=y$

Player 1's indifference:

 $0 y + 2(1-y) = 1 y + 0 (1-v) \rightarrow v=2/3$

Player 2's indifference:

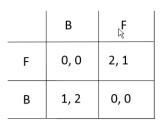
 $0 \times + 2(1-x) = 1 \times + 0 (1-x) \rightarrow x=2/3$

→ Mixed NE: ((2/3, 1/3); (2/3, 1/3))

- Another example: Game of chicken
- (same reasoning) >

Mixed NE: ((1/2, 1/2); (1/2, 1/2))

4)		



	Т	W
Т	-1,-1	2, 1
W	1, 2	0, 0

В

0, 0

1, 2

Т

F

В

F

2, 1

0, 0



Nash Equilibrium: More than one NE

- Another example: Battle of the sexes
- Two pure NE: (F;F) and (B;B)
- One mixed NE: Indifference condition
- \rightarrow Let $\sigma_1(F)=x$ and $\sigma_2(B)=y$

Player 1's indifference:

 $0 y + 2(1-y) = 1 y + 0 (1-y) \rightarrow y=2/3$

Player 2's indifference:

 $0 \times + 2(1-x) = 1 \times + 0 (1-x) \rightarrow x=2/3$

→ Mixed NE: ((2/3, 1/3); (2/3, 1/3))

- Another example: G
- (same reasoning) -

Mixed NE: ((1/2, 1/2)

Game of chicken		'	•
)	Т	-1,-1	2, 1
); (1/2, 1/2))		4.2	0.0
	W	1, 2	0, 0

• Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

- Another example: Battle of the sexes
- Two pure NE: (F:F) and (B:B)
- One mixed NE: Indifference condition \rightarrow Let $\sigma_1(F)=x$ and $\sigma_2(B)=y$

Player 1's indifference:

 $0 \text{ v} + 2(1-\text{v}) = 1 \text{ v} + 0 (1-\text{v}) \rightarrow \text{v} = 2/3$

Player 2's indifference:

 $0 \times + 2(1-x) = 1 \times + 0 (1-x) \rightarrow x=2/3$

→ Mixed NE: ((2/3, 1/3); (2/3, 1/3))

- Another example: Game of chicken
- (same reasoning) →

Mixed NE: ((1/2, 1/2); (1/2, 1/2))

	В	F
F	0,0	2, 1
В	1, 2	0, 0

	T	W
Т	-1,-1	2, 1
W	1, 2	0, 0

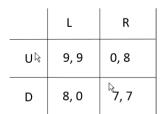
Sames in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

Risk Dominance / Pareto Optimality

- In this game: (Among others) two pure NE: (U,L) and (D,R); (U,L): Pareto dominates (D,R)
- But: For player 1 D is safer (guarrantees min payoff of 7) \rightarrow If p(R) > 1/8 don't go for (U,L) \rightarrow no certainty!
- Pregame-communication / agreement on (U,L)?!

No: player 2 gains if player 1 plays U → player 2 will always tell "L" regardless of true intentions → agreement is worthless









Nash Equilibrium: More than one NE

Risk Dominance / Pareto Optimality

- In this game: (Among others) two pure NE: (U,L) and (D,R); (U,L): Pareto dominates (D,R)
- But: For player 1 D is safer (guarrantees min payoff of 7) \rightarrow If p(R) > 1/8 don't go for (U,L) \rightarrow no certainty!
- Pregame-communication / agreement on (U,L) ?!

No: player 2 gains if player 1 plays U → player 2 will always tell "L" regardless of true intentions → agreement is worthless

	L	R
U	9, 9	0, 8
D	8, 0	7, 7

R

Sames in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

Risk Dominance / Pareto Optimality

- In this game: (Among others) two pure NE: (U,L) and (D,R); (U,L): Pareto dominates (D,R)
- But: For player 1 D is safer (guarrantees min payoff of 7) \rightarrow If p(R) > 1/8 don't go for (U,L) \rightarrow no certainty!
- Pregame-communication / agreement on (U,L) ?!

No: player 2 gains if player 1 plays U → player 2 will always tell "L" regardless of true intentions → agreement is worthless

	L	R
U	9, 9	0, 8
D	8, 0	7, 7

Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

Risk Dominance / Pareto Optimality

- In this game: (Among others) two pure NE: (U,L) and (D,R); (U,L): Pareto dominates (D,R)
- But: For player 1 D is safer (guarrantees min payoff of 7) \rightarrow If p(R) > 1/8 don't go for (U,L) \rightarrow no certainty!
- Pregame-communication / agreement on (U,L) ?!

No: player 2 gains if player 1 plays U → player 2 will always tell "L" regardless of true intentions → agreement is worthless

	L	R
U	9, 9	0, 8
D	8, 0	7, 7

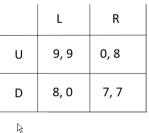
s in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

Risk Dominance / Pareto Optimality

- In this game: (Among others) two pure NE: (U,L) and (D,R); (U,L): Pareto dominates (D,R)
- But: For player 1 D is safer (guarrantees min payoff of 7) \rightarrow If p(R) > 1/8 don't go for (U,L) \rightarrow no certainty!
- Pregame-communication / agreement on (U,L) ?!

No: player 2 gains if player 1 plays U → player 2 will always tell "L" regardless of true intentions → agreement is worthless









Nash Equilibrium: More than one NE Risk Dominance / Pareto Optimality

	L	R
U	0,0,10	-5,-5,0
D	-5,-5,0	1,1,-5
A		

	L	R
U	-2,-2,0	-5,-5,0
D	-5,-5,0	-1,-1,5
	B	

- Three player game: Two pure NE: (U,L,A) and (D,R,B); (and one mixed); (U,L,A) pareto-dominates (D,R,B)
- If player 3's choice is fixed \rightarrow Two player game \rightarrow (D,R) is pareto-dominant \rightarrow if players 1 and 2 expect A: coordinate on (D,R).
- → concept of "coalition proof eq." (here (D,R,B))(see [1])



Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE Risk Dominance / Pareto Optimality

	L	R
U	0,0,10	-5,-5,0
D	-5,-5,0	1,1,-5
^		

	L	R
U	-2,-2,0	-5,-5,0
D	-5,-5,0	-1,-1,5
В		

- Three player game: Two pure NE: (U,L,A) and (D,R,B); (and one mixed); (U,L,A) pareto-dominates (D,R,B)
- If player 3's choice is fixed \rightarrow Two player game \rightarrow (D,R) is pareto-dominant \rightarrow if players 1 and 2 expect A: coordinate on (D,R).
- \rightarrow concept of "coalition proof eq." (here (D,R,B))(see [1])





Nash Equilibrium: More than one NE Risk Dominance / Pareto Optimality

	L	R
U	0,0,10	5,-5,0
D	-5,-5,0	1,1,-5
Λ		

	L	R
U	-2,-2,0	-5,-5,0
D	-5,-5,0	-1,-1,5
В		

- Three player game: Two pure NE: (U,L,A) and (D,R,B); (and one mixed); (U,L,A) pareto-dominates (D,R,B)
- If player 3's choice is fixed \rightarrow Two player game \rightarrow (D,R) is pareto-dominant \rightarrow if players 1 and 2 expect A: coordinate on (D,R).
- → concept of "coalition proof eq." (here (D,R,B))(see [1])

Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE Risk Dominance / Pareto Optimality

	L	R
U	0,0,10	-5,-5,0
D	-5,-5,0	1,1,-5
		1

	L	R
U	-2,-2,0	-5,-5,0
D	-5,-5,0	-1,-1,5
В		3

- Three player game: Two pure NE: (U,L,A) and (D,R,B); (and one mixed); (U,L,A) pareto-dominates (D,R,B)
- If player 3's choice is fixed \rightarrow Two player game \rightarrow (D,R) is pareto-dominant \rightarrow if players 1 and 2 expect A: coordinate on (D,R).
- \bullet concept of "coalition proof eq." (here (D,R,B))(see [1])



Summer in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE Risk Dominance / Pareto Optimality

	L	R
U	0,0,10	-5,-5,0
D	-5,-5,0	1,1,-5
Δ		

	L	R
U	-2,-2,0	-5,-5,0
D	-5,-5,0	-1,-1,5
В		

Three player game: Two pure NE: (U,L,A) and (D,R,B); (and one mixed); (U,L,A) pareto-dominates (D,R,B)

R

- If player 3's choice is fixed \rightarrow Two player game \rightarrow (D,R) is paretodominant \rightarrow if players 1 and 2 expect A: coordinate on (D,R).
- → concept of "coalition proof eq." (here (D,R,B))(see [1])



Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

Pure NE: One cell ->
For A: cell's payoff for A must be (weak)
maximum over rows in that column
For B: cell's payoff for B must be (weak)
maximum over column in that row

Example: (U,R) is	pure NE i	fa _{UR} ≥a _{DR}	and b _{UR} ≥
b_UL			

		L	R	
р	U	a _{UL} , b _{UL}	a _{UR} , b _{UR}	
Player A 1-p	D	a _{DL} , b _{DL}	a _{DR} , b _{DR}	

Player B

1-q

James in Strategic Form & Nash Equilibrium

Maria Laure Company Co		ı iay	/CI D	
Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])		q	1-q	
(366 [2])		L	R	
Pure NE: One cell →				
For A: cell's payoff for A must be (weak)	U	a _{UL} , b _{UL}	a _{UR} , b _{UR}	
maximum over row☆in that column Player A —				-
For B: cell's payoff for B must be (weak)	D	a _n , b _n	ass bss	

Player R

Player B

maximum over column in that row



James in Strategic Form & Nash Equilibrium

(see [2])			q	1 -q
(300 [2])			L	R
Pure NE: One cell > For A: cell's payoff for A must be (weak) maximum over rows in that column	p Plaver A –	U	a _{UL} , b _{UL}	a _{UR} , b _{UR}
For B: cell's payoff for B must be (weak) maximum over column in that row	1-p	D	a _{DL} , b _{DL}	a _{DR} , b _{DR}

Example: (U,R) is pure NE if $a_{UR} \ge a_{DR}$ and $b_{UR} \ge b_{UL}$ k



[•] Example: (U,R) is pure NE if $a_{UR} \ge a_{DR}$ and $b_{UR} \ge b_{III}$

Regional Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

for 2 x 2 Games	•	q	1 -q
		L	R
p Player A —	U	a _{UL} , b _{UL}	a _{UR} , b _{UR}
•	D	a _{DL} , b _{DL}	a _{DR} , b _{DR}

Player B

Player B

1-q

R

• If B plays mixed strategy (q, 1-q) or short q, A's best response is (p*(q), 1-p*(q)) or short p*(q).

\bullet u _A (p,q) = pq a _{UL} + p(1-q) a _{UR} + (1-p)q a _{DL} + (1-p)(1-q) a _{DR}	
$^{\downarrow}$ = p [(a_{UL}^{\uparrow} - a_{UR} - a_{DL} + a_{DR})q + a_{UR} - a_{DR}] + (a_{DL} - a_{DR})q + a_{DR}	R
= $p \delta(q) + (a_{DL} - a_{DR})q + a_{DR}$	

- Now: case differentiation w.r.t. sign of $\delta(q)$: $u_A(p,q)$ will be maximal for:
 - p*=1 if $\delta(q)$ is positive
 - p*=0 if $\delta(q)$ is negative
 - all p in [0,1] if $\delta(q)$ is zero

Sames in Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

Mixed NE. Start with player As view
If B plays mixed strategy (q, 1-q)
or short q, A's best response is

 $(p^*(q), 1-p^*(q))$ or short $p^*(q)$.

Mixed NE: Start with player Maying

W	р	U	a _{UL} , b _{UL}	a _{UR} , b _{UR}
Player A 1		D	a _{DL} , b _{DL}	a _{DR} , b _{DR}

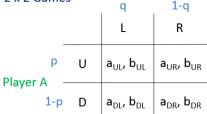
•
$$u_A(p,q) = pq a_{UL} + p(1-q) a_{UR} + (1-p)q a_{DL} + (1-p)(1-q) a_{DR}$$

= $p [(a_{UL} - a_{UR} - a_{DL} + a_{DR})q + a_{UR} - a_{DR}] + (a_{DL} - a_{DR})q + a_{DR}$
= $p \delta(q) + (a_{DL} - a_{DR})q + a_{DR}$

- Now: case differentiation w.r.t. sign of $\delta(q)$: $u_A(p,q)$ will be maximal for:
 - p*=1 if $\delta(q)$ is positive
 - p*=0 if $\delta(q)$ is negative
 - all p in [0,1] if $\delta(q)$ is zero

ames in Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])



Player B

- Mixed NE: Start with player A's view
- If B plays mixed strategy (q, 1-q) or short q, A's best response is (p*(q), 1-p*(q)) or short p*(q).
- $\begin{aligned} u_{A}(p,q) &= pq \ a_{UL} + p(1-q) \ a_{UR} + (1-p)q \ a_{DL} + (1-p)(1-q) \ a_{DR} \\ &= p \ [(a_{UL} a_{UR} a_{DL} + a_{DR})q + a_{UR} a_{DR}] \ + (a_{DL} a_{DR})q + a_{DR} \\ &= p \ [\delta(q) + (a_{DL} a_{DR})q + a_{DR} \end{aligned}$
- Now: case differentiation w.r.t. sign of $\delta(q)$: $u_A(p,q)$ will be maximal for:
 - $^{\bullet}$ p*=1 if δ (q) is positive
 - p*=0 if $\delta(q)$ is negative
 - all p in [0,1] if $\delta(q)$ is zero

James in Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

$$u_A(p,q) = p \delta(q) + (a_{DL} - a_{DR})q + a_{DR}$$

- Further case differentiation:
 - (1) If $\delta(q)=0$ for all $q \to f$ for all q, A's best response are all p in [0,1]. (\to A is completely indifferent)
 - If δ is not identically zero: $\delta(q)$ is affine linear: Has at most one zero q^+ with $\delta(q^+)=0$. (exactly one if we set for $q^+=\infty$ for $(a_{UL}-a_{UR}-a_{DI}+a_{DR})=0)$ \rightarrow
 - (2) If $q^+\notin [0,1]$: A plays $p^* \stackrel{\square}{=} 1$ or $p^*=0$ depending on sign of $\delta(q)$. (\rightarrow A has a dominant pure strategy)
 - (3) If $q^+ \in [0,1]$: A plays $p^* = 1$ or $p^* = 0$ if $\delta(q) \neq 0$ (depending on sign of $\delta(q)$) and can play any p if $q = q^+$. (\rightarrow A plays strategically)



Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

$$u_A(p,q) = p \delta(q) + (a_{DL} - a_{DR})q + a_{DR}$$

- Further case differentiation:
 - (1) If $\delta(q)=0$ for all $q \to 0$ for all q, A's best response are all p in [0,1]. ($\to 0$ A is completely indifferent)
 - If δ is not identically zero: $\delta(q)$ is affine linear: Has at most one zero q^+ with $\delta(q^+)=0$. (exactly one if we set for $q^+=\infty$ for $(a_{UL}-a_{UR}-a_{DI}+a_{DR})=0)$ \rightarrow
 - (2) If $q^+\notin [0,1]$: A plays $p^*=1$ or $p^*=0$ depending on sign of $\delta(q)$. (\rightarrow A has a dominant pure strategy)
 - (3) If $q^+ \in [0,1]$: A plays $p^* = 1$ or $p^* = 0$ if $\delta(q) \neq 0$ (depending on sign of $\delta(q)$) and can play any p if $q = q^+$. (\rightarrow A plays strategically)



Simes in Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

$$u_A(p,q) = p \delta(q) + (a_{DL} - a_{DR})q + a_{DR}$$

- Further case differentiation:
 - (1) If $\delta(q)=0$ for all $q \to f$ for all q, A's best response are all p in [0,1]. ($\to A$ is completely indifferent)
 - If δ is not identically zero: $\delta(q)$ is affine linear: Has at most one zero q^+ with $\delta(q^+)=0$. (exactly one if we set for $q^+=\infty$ for $(a_{UL}-a_{UR}-a_{DI}+a_{DR})=0)$
 - (2) If $q^+\notin [0,1]$: A plays $p^*=1$ or $p^*=0$ depending on sign of $\delta(q)$. (\rightarrow A has a dominant pure strategy)
 - (3) If $q^+ \in [0,1]$: A plays $p^* = 1$ or $p^* = 0$ if $\delta(q) \neq 0$ (depending on sign of $\delta(q)$) and can play any p if $q = q^+$. (\rightarrow A plays strategically)



Sames in Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

Juine		q	1-q
		L	R
p ^	U	a _{UL} , b _{UL}	a _{UR} , b _{UR}
1-p	D	a _{DL} , b _{DL}	a _{DR} , b _{DR}

Plaver B

- Mixed NE: Start with player A's view
- If B plays mixed strategy (q, 1-q) or short q, A's best response is (p*(q), 1-p*(q)) or short p*(q).

Player

- Now: case differentiation w.r.t. sign of $\delta(q)$: $u_A(p,q)$ will be maximal for:
 - $^{\bullet}$ p*=1 if δ (q) is positive
 - p*=0 if $\delta(q)$ is negative
 - all p in [0,1] if δ(q) is zero

Sames in Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

$$u_A(p,q) = p \delta(q) + (a_{DL} - a_{DR})q + a_{DR}$$

- Further case differentiation:
 - (1) If $\delta(q)=0$ for all $q \to 0$ for all q, A's best response are all p in [0,1]. ($\to 0$ A is completely indifferent)
 - If δ is not identically zero: $\delta(q)$ is affine linear: Has at most one zero q^+ with $\delta(q^+)=0$. (exactly one if we set for $q^+=\infty$ for $(a_{UL}-a_{UR}-a_{DI}+a_{DR})=0)$ \rightarrow
 - (2) If $q^+\notin [0,1]$: A plays $p^*=1$ or $p^*=0$ depending on sign of $\delta(q)$. (\rightarrow A has a dominant pure strategy)
 - (3) If $q^+ \in [0,1]$: A plays $p^* = 1$ or $p^* = 0$ if $\delta(q) \neq 0$ (depending on sign of $\delta(q)$) and can play any p if $q = q^+$. (\rightarrow A plays strategically)

Regional Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

	q	1-q	
	L	R	
U	a _{UL} , b _{UL}	a _{UR} , b _{UR}	
D	as, bs.	ass has	

Player B

- Mixed NE: Start with player A's view
- If B plays mixed strategy (q, 1-q) or short q, A's best response is (p*(q), 1-p*(q)) or short p*(q).
- $u_A(p,q) = pq a_{UL} + p(1-q) a_{UR} + (1-p)q a_{DL} + (1-p)(1-q) a_{DR}$ = $p [(a_{UL} - a_{UR} - a_{DL} + a_{DR})q + a_{UR} - a_{DR}] + (a_{DL} - a_{DR})q + a_{DR}$ = $p \delta(q) + (a_{DL} - a_{DR})q + a_{DR}$

Player A

- Now: case differentiation w.r.t. sign of $\delta(q)$: $u_A(p,q)$ will be maximal for:
 - p*=1 if $\delta(q)$ is positive
 - p*=0 if $\delta(q)$ is negative
 - all p in [0,1] if $\delta(q)$ is zero



Sames in Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

$$u_A(p,q) = p \delta(q) + (a_{DL} - a_{DR})q + a_{DR}$$

- Further case differentiation:
 - (1) If $\delta(q)=0$ for all $q \to f$ for all q, A's best response are all p in [0,1]. ($\to A$ is completely indifferent)
 - If δ is not identically zero: $\delta(q)$ is affine linear: Has at most one zero q^+ with $\delta(q^+)=0$. (exactly one if we set for $q^+=\infty$ for $(a_{UL}-a_{UR}-a_{DI}+a_{DR})=0)$ \rightarrow
 - (2) If $q^+\notin [0,1]$: A plays $p^*=1$ or $p^*=0$ depending on sign of $\delta(q)$. (\rightarrow A has a dominant pure strategy)
 - (3) If $q^+ \in [0,1]$: A plays $p^* = 1$ or $p^* = 0$ if $\delta(q) \neq 0$ (depending on sign of $\delta(q)$) and can play any p if $q = q^+$. (\rightarrow A plays strategically)



Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

$$u_A(p,q) = p \delta(q) + (a_{DL} - a_{DR})q + a_{DR}$$

- Further case differentiation:
 - (1) If $\delta(q)=0$ for all $q \to f$ for all q, A's best response are all p in [0,1]. ($\to A$ is completely indifferent)
 - If δ is not identically zero: $\delta(q)$ is affine linear: Has at most one zero q^+ with $\delta(q^+)=0$. (exactly one if we set for $q^+=\infty$ for $(a_{UL}-a_{UR}-a_{DL}+a_{DR})=0)$ \rightarrow
 - (2) If $q^+\notin [0,1]$: A plays $p^*=1$ or $p^*=0$ depending on sign of $\delta(q)$. (\rightarrow A has a dominant pure strategy)
 - (3) If $q^+ \in [0,1]$: A plays $p^* = 1$ or $p^* = 0$ if $\delta(q) \neq 0$ (depending on sign of $\delta(q)$) and can play any p if $q = q^+$. (\rightarrow A plays strategically)

(1) (b) (2) (B) (Q) (...)

Sames in Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

$$u_A(p,q) = p \delta(q) + (a_{DL} - a_{DR})q + a_{DR}$$

- Further case differentiation:
 - (1) If $\delta(q)=0$ for all $q \to f$ for all q, A's best response are all p in [0,1]. (\to A is completely indifferent)
 - If δ is not identically zero: $\delta(q)$ is affine linear: Has at most one zero q^+ with $\delta(q^+)=0$. (exactly one if we set for $q^+=\infty$ for $(a_{UL}-a_{UR}-a_{DI}+a_{DR})=0)$ \rightarrow
 - (2) If $q^+\notin [0,1]$: A plays $p^*=1$ or $p^*=0$ depending on sign of $\delta(q)$. (\rightarrow A has a dominant pure strategy)
 - (3) If $q^+ \in [0,1]$: A plays $p^* = 1$ or $p^* = 0$ if $\delta(q) \neq 0$ (depending on sign of $\delta(q)$) and can play any p if $q = q^+$. (\rightarrow A plays strategically)

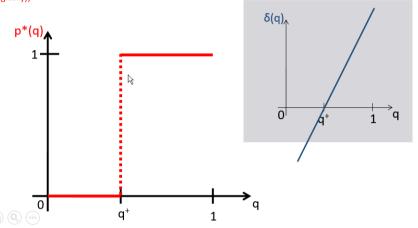


Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

 $u_A(p,q) = p \delta(q) + (a_{DL} - a_{DR})q + a_{DR}$

Vizualization of case (3)

(If $q^* \in [0,1]$: A plays $p^* = 1$ or $p^* = 0$ if $\delta(q) \neq 0$ (depending on sign of $\delta(q)$) and can play any p if $q = q^*$. (\Rightarrow A plays strategically))



Sames in Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

Games			7	. 1	L-q
		ı	-		R
- n					

Player B

• Now: Analysis for B:

• If A plays mixed strategy (p, 1-p) or short p, B's best response is (q*(p), 1-q*(p)) or short q*(p).

p Player A – 1-p	U	a _{UL} , b _{UL}	a _{UR} , b _{UR}
	D	a _{DL} , b _{DL}	a _{DR} , b _{DR}

- $u_B(p,q) = q [(b_{UL} b_{UR} b_{DL} + b_{DR})p + b_{DL} b_{DR}] + (b_{UR} b_{DR})p + b_{DR}$ $= q \gamma(p) + (b_{UR} b_{DR})p + b_{DR}$
- Exactly the same case differentiations as in case of A
- Trivially: Nash-Equilibrium iff q*(p) = p*(q).

Sames in Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

		L	R
p layer A — 1-p	U	a _{UL} , b _{UL}	a _{UR} , b _{UR}
	D	a _{DL} , b _{DL}	a _{DR} , b _{DR}

Player B

1-a

- Now: Analysis for B:
- If A plays mixed strategy (p, 1-p) or short p, B's best response is (q*(p), 1-q*(p)) or short q*(p).
- $u_B(p,q) = q [(b_{UL} b_{UR} b_{DL} + b_{DR})p + b_{DL} b_{DR}] + (b_{UR} b_{DR})p + b_{DR}$ $= q \gamma(p) + (b_{UR} b_{DR})p + b_{DR}$
- Exactly the same case differentiations as in case of A
- Trivially: Nash-Equilibrium iff q*(p) = p*(q).



Stage Games with Observed Actions

Preliminaries

- In part1: Players made choices simultaneously
- Now: order of choices makes knowledge explicit → Multi-Stage
 Games with Observed Actions
- **Example: Stackelberg equilibrium in Cournot model:** Player1 chooses output level q_1 first, player 2 observes q_1 and then chooses q_2 .
- Assume costless production and linear demand \Rightarrow $u_i(q_1, q_2)=[12-q_1-q_2] q_i$
- Player 2's reaction on q_1 will be $s_2(q_1) \rightarrow u_i(q_1, q_2) = u_i(q_1, s_2(q_1))$
- Pure NE: Neither player can gain by switching from NE to other pure strategy:
 - •(1) For given q_1 : Maximize: $\max_{q_2} u_2(q_1, q_2) \rightarrow \text{we get Cournot}$ reaction $r_2(q_1)=6 - q_1/2$; Now: Maximzie: $\max_{q_1} u_1(q_1, r_2(q_1)) \rightarrow q^*_1$ =6 \rightarrow NE (Stackelberg EQ): $(q^*_1, r_2(q^*_1))$



Output levels q are chosen from sets Q

• Cost of production is c_i(q_i)

• Market price is $p(q) = p(q_1+q_2)_{\triangleright}$

• Firm i's profit is then $u_i(q_1, q_2) = q_i p(q) - c_i(q_i)$

• Cournot reaction functions $r_1: Q_2 \rightarrow Q_1$ and $r_2: Q_1 \rightarrow Q_2$ specify optimal reaction on output level of opponent

ames in Strategic Form & Nash Equilibrium

Nash Equilibrium: Example: Cournot Competition

Cournot model: Duopoly. Each of two firms (players) i produces same good.

Output levels q_i are chosen from sets Q_i

Cost of production is c_i(q_i)

• Market price is $p(q) = p(q_1+q_2)$

• Firm i's profit is then $u_i(q_1, q_2) = q_i p(q) - c_i(q_i)$

• Cournot reaction functions $r_1: Q_2 \rightarrow Q_1$ and $r_2: Q_1 \rightarrow Q_2$ specify optimal reaction on output level of opponent

Nash Equilibrium: Example: Cournot Competition

 Cournot model: Duopoly. Each of two firms (players) i produces same good.

Output levels q are chosen from sets Q

Cost of production is c_i(q_i)

• Market price is $p(q) = p(q_1+q_2)$

• Firm i's profit is then $u_i(q_1, q_2) = q_i p(q) - c_i(q_i)$

• Cournot reaction functions $r_1: Q_2 \rightarrow Q_1$ and $r_2: Q_1 \rightarrow Q_2$ specify optimal reaction on output level of opponent

Strategic Form & Nash Equilibrium

Nash Equilibrium: Example: Cournot Competition

Under certain reasonable assumptions (see [1]) we can maximize e.g. $u_2(q_1, q_2)$ by solving $\partial/\partial q_2$ $u_2(q_1, q_2) = 0$ which yields $\partial/\partial q_2$ $[q_2 p(q_1, q_2) - c_2(q_2)] = p(q_1, q_2) + p'(q_1, q_2) q_2 - c_2'(q_2) = 0$.

Inserting r_2 (q_1) for q_2

 $p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) r_2(q_1) - c_2'(r_2(q_1)) = 0$

gives the defining equation for $\boldsymbol{r}_{2}\left(.\right)$.

(analogous for r_1 (.)).

• The intersections of the functions r_2 and r_1 are the NE of the Cournot game.

Example: Linear demand p(q) = max(0, 1-q); linear cost: $c_i(q_i) = c q_i$:

 \rightarrow r₂ (q₁) =1/2 (1- q₁ -c); r₁ (q₂) =1/2 (1- q₂ -c);

 \rightarrow NE: $q_2^* = r_2 (q_1^*) = 1/3 (1-c) = q_1^* = r_1 (q_2^*)$



Nash Equilibrium: Example: Cournot Competition

Under certain reasonable assumptions (see [1]) we can maximize e.g. $u_2(q_1, q_2)$ by solving $\partial/\partial q_2$ $u_2(q_1, q_2) = 0$ which yields

$$\partial/\partial q_2 [q_2 p(q_1,q_2) - c_2(q_2)] = p(q_1,q_2) + p'(q_1,q_2) q_2 - c_2'(q_2) = 0.$$

Inserting r₂ (q₁) for q₂

$$p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) r_2(q_1) - c_2'(r_2(q_1)) = 0$$

gives the defining equation for r_2 (.)

(analogous for r₁ (.)).

- The intersections of the functions r_2 and r_1 are the NE of the Cournot game.
- Example: Linear demand p(q) = max(0, 1-q); linear cost: $c_i(q_i) = c_iq_i$:

$$\rightarrow$$
 r₂ (q₁) =1/2 (1- q₁ -c); r₁ (q₂) =1/2 (1- q₂ -c);

$$\rightarrow$$
 NE: $q_2^* = r_2 (q_1^*) = 1/3 (1-c) = q_1^* = r_1 (q_2^*)$

Strategic Form & Nash Equilibrium

Nash Equilibrium: Example: Cournot Competition

Under certain reasonable assumptions (see [1]) we can maximize e.g. $u_2(q_1, q_2)$ by solving $\partial/\partial q_2$ $u_2(q_1, q_2) = 0$ which yields

$$\partial/\partial q_2 [q_2 p(q_1,q_2) - c_2(q_2)] = p(q_1,q_2) + p'(q_1,q_2) q_2 - c_2'(q_2) = 0.$$

Inserting r_2 (q_1) for q_2

$$p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) r_2(q_1) - c_2'(r_2(q_1)) = 0$$

gives the defining equation for r_2 (.).

(analogous for r_1 (.)).

- The intersections of the functions r_2 and r_1 are the NE of the Cournot game.
- Example: Linear demand p(q) = max(0, 1-q); linear cost: $c_i(q_i) = c q_i$:

$$\rightarrow$$
 r₂ (q₁) =1/2 (1- q₁ -c); r₁ (q₂) =1/2 (1- q₂ -c);

$$\rightarrow$$
 NE: $q_2^* = r_2 (q_1^*) = 1/3 (1-c) = q_1^* = r_1 (q_2^*)$



Nash Equilibrium: Example: Cournot Competition

• Under certain reasonable assumptions (see [1]) we can maximize e.g. $u_2(q_1, q_2)$ by solving $\partial/\partial q_2$ $u_2(q_1, q_2) = 0$ which yields

$$\partial/\partial q_2 [q_2 p(q_1,q_2) - c_2(q_2)] = p(q_1,q_2) + p'(q_1,q_2) q_2 - c_2'(q_2) = 0.$$

Inserting r₂ (q₁) for q₂

$$p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) r_2(q_1) - c_2'(r_2(q_1)) = 0$$

gives the defining equation for r_2 (.).

(analogous for r₁ (.)).

- The intersections of the functions r_2 and r_1 are the NE of the Cournot game.
- Example: Linear demand p(q) = max(0, 1-q); linear cost: c_i(q_i) = c q_i:

$$\rightarrow$$
 r₂ (q₁) =1/2 (1- q₁ -c); r₁ (q₂) =1/2 (1- q₂ -c);

$$\rightarrow$$
 NE: $q_2^* = r_2 (q_1^*) = 1/3 (1-c) = q_1^* = r_1 (q_2^*)$

Strategic Form & Nash Equilibrium

Nash Equilibrium: Example: Cournot Competition

• Under certain reasonable assumptions (see [1]) we can maximize e.g. $u_2(q_1, q_2)$ by solving $\partial/\partial q_2$ $u_2(q_1, q_2) = 0$ which yields

$$\partial/\partial q_2 [q_2 p(q_1,q_2) - c_2(q_2)] = p(q_1,q_2) + p'(q_1,q_2) q_2 - c_2'(q_2) = 0.$$

Inserting r_2 (q_1) for q_2

$$p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) r_2(q_1) - c_2'(r_2(q_1)) = 0$$

gives the defining equation for r_2 (.)

(analogous for r_1 (.)).

- The intersections of the functions r_2 and r_1 are the NE of the Cournot game.
- Example: Linear demand p(q) = max(0, 1-q); linear cost: $c_i(q_i) = c q_i$:

$$\rightarrow$$
 r₂ (q₁) =1/2 (1- q₁ -c); r₁ (q₂) =1/2 (1- q₂ -c);

$$\rightarrow$$
 NE: $q_2^* = r_2 (q_1^*) = 1/3 (1-c) = q_1^* = r_1 (q_2^*)$



Nash Equilibrium: Example: Cournot Competition

Under certain reasonable assumptions (see [1]) we can maximize e.g. $u_2(q_1, q_2)$ by solving $\partial/\partial q_2$ $u_2(q_1, q_2) = 0$ which yields

$$\partial/\partial q_2 [q_2 p(q_1,q_2) - c_2(q_2)] = p(q_1,q_2) + p'(q_1,q_2) q_2 - c_2'(q_2) = 0.$$

Inserting r₂ (q₁) for q₂

$$p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) r_2(q_1) - c_2'(r_2(q_1)) = 0$$

gives the defining equation for r_2 (.)

(analogous for r_1 (.)).

- The intersections of the functions r_2 and r_1 are the NE of the Cournot game.
- Example: Linear demand p(q) = max(0, 1-q); linear cost: c_i(q_i) = c q_i:

$$\rightarrow$$
 r₂ (q₁) =1/2 (1- q₁ -c); r₁ (q₂) =1/2 (1- q₂ -c);

$$\rightarrow$$
 NE: $q_2^* = r_2 (q_1^*) = 1/3 (1-c) = q_1^* = r_1 (q_2^*)$



Preliminaries

- In part1: Players made choices simultaneously
- Now: order of choices makes knowledge explicit → Multi-Stage Games with Observed Actions
- **Example: Stackelberg equilibrium in Cournot model: Player1 chooses** output level q_1 first, player 2 observes q_1 and then chooses q_2 .
- Assume costless production and linear demand \Rightarrow $u_i(q_1, q_2)=[12-q_1-q_2] q_i$
- Player 2's reaction on q_1 will be $s_2(q_1) \rightarrow u_i(q_1, q_2) = u_i(q_1, s_2(q_1))$
- Pure NE: Neither player can gain by switching from NE to other pure strategy:
 - •(1) For given q_1 : Maximize: $\max_{q_2} u_2(q_1, q_2) \rightarrow \text{we get Cournot}$ reaction $r_2(q_1)=6 - q_1/2$; Now: Maximzie: $\max_{q_1} u_1(q_1, r_2(q_1)) \rightarrow q^*_1$ =6 \rightarrow NE (Stackelberg EQ): $(q^*_1, r_2(q^*_1))$

Strategic Form & Nash Equilibrium

Nash Equilibrium: Example: Cournot Competition

- Under certain reasonable assumptions (see [1]) we can maximize e.g. $u_2(q_1, q_2)$ by solving $\partial/\partial q_2$ $u_2(q_1, q_2) = 0$ which yields $\partial/\partial q_2$ $[q_2 p(q_1, q_2) c_2(q_2)] = p(q_1, q_2) + p'(q_1, q_2) q_2 c_2'(q_2) = 0$. Inserting r_2 (q_1) for q_2 $p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) r_2(q_1) c_2'(r_2(q_1)) = 0$ gives the defining equation for r_2 (.) . $\[\]$ (analogous for r_1 (.)).
- $^{\bullet}$ The intersections of the functions r_2 and r_1 are the NE of the Cournot game.
- Example: Linear demand p(q) = max(0, 1-q); linear cost: $c_i(q_i) = c q_i$:

$$\rightarrow$$
 r₂ (q₁) =1/2 (1- q₁ -c); r₁ (q₂) =1/2 (1- q₂ -c);

$$\rightarrow$$
 NE: $q_2^* = r_2 (q_1^*) = 1/3 (1-c) = q_1^* = r_1 (q_2^*)$

Stage Games with Observed Actions

Preliminaries

- In part1: Players made choices simultaneously
- Now: order of choices makes knowledge explicit → Multi-Stage Games with Observed Actions
- Example: Stackelberg equilibrium in Cournot model: Player1 chooses output level q_1 first, player 2 observes q_1 and then chooses q_2 .
- Assume costless production and linear demand \rightarrow $u_i(q_1, q_2)=[12-q_1-q_2] q_i$
- Player 2's reaction on q_1 will be $s_2(q_1) \rightarrow u_i(q_1, q_2) = u_i(q_1, s_2(q_1))$
- Pure NE: Neither player can gain by switching from NE to other pure strategy:
 - •(1) For given q_1 : Maximize: $\max_{q_2} u_2(q_1, q_2) \rightarrow \text{we get Cournot}$ reaction $r_2(q_1)=6 q_1/2$; Now: Maximize: $\max_{q_1} u_1(q_1, r_2(q_1)) \rightarrow q^*_1 = 6 \rightarrow \text{NE (Stackelberg EQ): } (q^*_1, r_2(q^*_1))$

- Stage Games with Observed Actions

Preliminaries

- In part1: Players made choices simultaneously
- Now: order of choices makes knowledge explicit → Multi-Stage Games with Observed Actions
- Example: Stackelberg equilibrium in Cournot model: Player1 chooses output level q₁ first, player 2 observes q₁ and then chooses q₂.
- Assume costless production and linear demand \rightarrow $u_i(q_1, q_2)=[12-q_1-q_2] q_i$
- Player 2's reaction on q_1 will be $s_2(q_1) \rightarrow u_i(q_1, q_2) = u_i(q_1, s_2(q_1))$
- Pure NE: Neither player can gain by switching from NE to other pure strategy:
- •(1) For given q_1 : Maximize: $\max_{q_2} u_2(q_1, q_2) \rightarrow \text{we get Cournot}$ reaction $r_2(q_1)=6-q_1/2$; Now: Maximzie: $\max_{q_1} u_1(q_1, r_2(q_1)) \rightarrow q^*_1$ =6 \rightarrow NE (Stackelberg EQ): $(q^*_1, r_2(q^*_1))$



Preliminaries

- Other NE: Cournot Equilibrium (from simultaneous play) $(q_1^c, q_2^c)=(4; 4);$
- Why is this a NE? If player 2 plays q^c_2 (independent of q_1), player 1 must play $r_1(q^c_2) = q^c_1$ in order to maximize u_1 . If player 1 plays q^c_1 , player 2 can play any s_2 with $s_2(q^c_1) = q^c_2$ in order to maximize u_2 , including s_2 = const = q^c_2 .
- Stackelberg EQ (q_1^* ; $r_2(q_1^*)$): payoff is (18;9); Cournot EQ (q_1^c ; q_2^c): payoff is (16;16);
- But: "player 2 plays q_2^c (independent of q_1)" is empty threat: If player 1 plays $q_1 \neq q_1^c$, player 2 is better of with q_2^c





Preliminaries

- In part1: Players made choices simultaneously
- ullet Now: order of choices makes knowledge explicit ightarrow Multi-Stage Games with Observed Actions
- Example: Stackelberg equilibrium in Cournot model: Player1 chooses output level q₁ first, player 2 observes q₁ and then chooses q₂.
- Assume costless production and linear demand \rightarrow $u_i(q_1, q_2)=[12-q_1-q_2] q_i$
- Player 2's reaction on q_1 will be $s_2(q_1) \rightarrow u_i(q_1, q_2) = u_i(q_1, s_2(q_1))$
- Pure NE: Neither player can gain by switching from NE to other pure strategy:
 - •(1) For given q_1 : Maximize: $\max_{q_2} u_2(q_1, q_2) \rightarrow \text{we get Cournot}$ reaction $r_2(q_1)=6 q_1/2$; Now: Maximzie: $\max_{q_1} u_1(q_1, r_2(q_1)) \rightarrow q_1^*$ $= 6 \rightarrow \text{NE}$ (Stackelberg EQ): $(q_1^*, r_2(q_1^*))$



Preliminaries

- Other NE: Cournot Equilibrium (from simultaneous play) $(q_1^c; q_2^c)=(4; 4);$
- Why is this a NE? If player 2 plays q^c_2 (independent of q_1), player 1 must play $r_1(q^c_2) = q^c_1$ in order to maximize u_1 . If player 1 plays q^c_1 , player 2 can play any s_2 with $s_2(q^c_1) = q^c_2$ in order to maximize u_2 , including s_2 = const = q^c_2 .
- Stackelberg EQ (q_1^* ; $r_2(q_1^*)$): payoff is (18;9); Cournot EQ (q_1^c ; q_2^c): payoff is (16;16);
- But: "player 2 plays q_2^c (independent of q_1)" is empty threat: If player 1 plays $q_1 \neq q_1^c$, player 2 is better of with $q_2(q_1)$

Stage Games with Observed Actions

Preliminaries

- Other NE: Cournot Equilibrium (from simultaneous play) $(q_1^c; q_2^c)=(4; 4)$;
- Why is this a NE? If player 2 plays q^c_2 (independent of q_1), player 1 must play $r_1(q^c_2) = q^c_1$ in order to maximize u_1 . If player 1 plays q^c_1 , player 2 can play any s_2 with $s_2(q^c_1) = q^c_2$ in order to maximize u_2 , including $s_2 = \text{const} = q^c_2$.
- Stackelberg EQ $(q_1^*; r_2(q_1^*))$: payoff is (18;9); Cournot EQ $(q_1^c; q_2^c)$: payoff is (16;16);
- But: "player 2 plays q^c_2 (independent of q_1)" is empty threat: If player 1 plays $q_1 \neq q^c_1$, player 2 is better of with $r_2(q_1)$

Preliminaries

- Other NE: Cournot Equilibrium (from simultaneous play) $(q_1^c; q_2^c)=(4; 4);$
- Why is this a NE? If player 2 plays q_2^c (independent of q_1), player 1 must play $r_1(q_2^c) = q_1^c$ in order to maximize u_1 . If player 1 plays q_1^c , player 2 can play any s_2 with $s_2(q_1^c) = q_2^c$ in order to maximize u_2 , including $s_2 = \text{const} = q_2^c$.
- Stackelberg EQ $(q_1^*; r_2(q_1^*))$: payoff is (18;9); Cournot EQ $(q_1^c; q_2^c)$: payoff is (16;16);
- But: "player 2 plays q_2^c (independent of q_1)" is empty threat: If player 1 plays $q_1 \neq q_1^c$, player 2 is better of with $q_2(q_1)$



- Stage Games with Observed Actions

Preliminaries

- Other NE: Cournot Equilibrium (from simultaneous play) $(q_1^c, q_2^c)=(4; 4)$;
- Why is this a NE? If player 2 plays q_2^c (independent of q_1), player 1 must play $r_1(q_2^c) = q_1^c$ in order to maximize u_1 . If player 1 plays q_1^c , player 2 can play any s_2 with $s_2(q_1^c) = q_2^c$ in order to maximize u_2 , including $s_2 = \text{const} = q_2^c$.
- Stackelberg EQ (q_1^* ; $r_2(q_1^*)$): payoff is (18;9); Cournot EQ (q_1^c ; q_2^c): payoff is (16;16);
- But: "player 2 plays q_2^c (independent of q_1)" is empty threat: If player 1 plays $q_1 \neq q_1^c$, player 2 is better of with $r_2(q_1)$

Stage Games with Observed Actions

Multistage Games

- Stages (0,...,K); At each stage k: all players choose actions simultaneously; All players know actions from all players from all previous stages
- Action profile of stage k: $a^k = (a_1^k, a_2^k, ..., a_l^k)$

De.

- History at end of stage k / beginning of stage k+1: $h^{k+1} = (a^0, a^1, \dots, a^k)$: $h^0 = \emptyset$
- Set of possible actions of player i if history hk : A_i(hk)
- Possible: $A_i(h^k)=\{$, do nothing" $\} \rightarrow$ Previous example and generally games of "alternating choice" included

Stage Games with Observed Actions

Multistage Games

- Stages (0,...,K); At each stage k: all players choose actions simultaneously; All players know actions from all players from all previous stages
- Action profile of stage k: $a^k = (a^k_1, a^k_2, ..., a^k_l)$
- History at end of stage k / beginning of stage k+1: $h^{k+1} = (a_0^0, a_1^1, ..., a^k); \quad h^0 = \emptyset$
- Set of possible actions of player i if history hk : A_i(hk)
- Possible: $A_i(h^k)=\{$, do nothing" $\} \rightarrow$ Previous example and generally games of "alternating choice" included

Multistage Games

- Stages (0,...,K); At each stage k: all players choose actions simultaneously; All players know actions from all players from all previous stages
- Action profile of stage k: $a^k = (a^k_1, a^k_2, ..., a^k_l)$
- History at end of stage k / beginning of stage k+1: $h^{k+1} = (a_0^k, a^1, ..., a^k); h^0 = \emptyset$
- Set of possible actions of player i if history hk: A_i(hk)
- Possible: $A_i(h^k) = \{\text{"do nothing"}\} \rightarrow \text{Previous example and generally games of "alternating choice" included}$



- Stage Games with Observed Actions

Multistage Games

- Pure strategy for player i is contingent plan $s_i = (s_i^0, s_i^1, ..., s_i^K)$ where the s_i^k map the history to set of possible actions at each stage: $s_i^k(h^k) \in A_i(h^k)$;
- Mixed strategy $\sigma_i^k(h^k)$: define prob.-distribution over $A_i(h^k)$
- Thus: $a^k = s^k (h^k) = s^k (a^0, a^2, ..., a^{k-1})$;
- (a⁰ .a² a^j) is called path
- Overall Payoff of player i: u_i(h^{K+1})

Pure strategy NE: Strategy profile s so that for all players i: $u_i(s_i, s_i) \ge u_i(s_i', s_i)$ for all s_i'



Multistage Games

- Pure strategy for player i is contingent plan $s_i = (s_i^0, s_i^1, ..., s_i^K)$ where the s_i^k map the history to set of possible actions at each stage: $s_i^k(h^k) \in A_i(h^k)$;
- Mixed strategy $\sigma_i^k(h^k)$: define prob.-distribution over $A_i(h^k)$
- Thus: $a^k = s^k(h^k) = s^k(a^0, a^2, ..., a^{k-1})$;
- (a⁰,a²,..., a^j) is called path
- Overall Payoff of player i: u_i(h^{K+1})

Pure strategy NE: Strategy profile s so that for all players i: $u_i(s_i, s_i) \ge u_i(s_i', s_i)$ for all s_i'

Multistage Games

- Pure strategy for player i is contingent plan $s_i = (s_i^0, s_i^1, ..., s_i^K)$ where the s_i^k map the history to set of possible actions at each stage: $s_i^k(h^k) \in A_i(h^k)$;
- Mixed strategy $\sigma_{i_k}^k(h^k)$: define prob.-distribution over $A_i(h^k)$
- Thus: $a^k = s^k (h^k) = s^k (a^0, a^2, ..., a^{k-1})$;
- (a⁰,a²,..., a^j) is called path
- Overall Payoff of player i: u_i(h^{K+1})

Pure strategy NE: Strategy profile s so that for all players i: $u_i(s_i, s_{ij}) \ge u_i(s_i', s_{ij})$ for all s_i'

Multistage Games

- Pure strategy for player i is contingent plan $s_i = (s_i^0, s_i^1, ..., s_i^K)$ where the s_i^k map the history to set of possible actions at each stage: $s_i^k(h^k) \in A_i(h^k)$;
- Mixed strategy $\sigma_i^k(h^k)$: define prob.-distribution over $A_i(h^k)$
- Thus: $a^k = s^k(h^k) = s^k(a^0, a^2, ..., a^{k-1})$;
- (a⁰,a²,..., a^j) is called path
- Overall Payoff of player i: $u_i(h^{K+1})$

Pure strategy NE: Strategy profile s so that for all players i: $u_i(s_i, s_{-i}) \ge u_i(s_i', s_{-i})$ for all s_i'





Multistage Games

- Pure strategy for player i is contingent plan $s_i = (s_i^0, s_i^1, ..., s_i^K)$ where the s_i^k map the history to set of possible actions at each stage: s_i^k (h_i^k) $\in A_i(h_i^k)$;
- Mixed strategy $\sigma_i^k(h^k)$: define prob.-distribution over $A_i(h^k)$
- Thus: $a^k = s^k(h^k) = s^k(a^0, a^2, ..., a^{k-1})$;
- (a⁰ .a² a^j) is called path
- Overall Payoff of player i: u_i(h^{K+1})

Pure strategy NE: Strategy profile s so that for all players i: $u_i(s_i, s_{-i}) \ge u_i(s'_i, s_{-i})$ for all s'_i



- Stackelberg EQ example: Alternating moves: backward induction easy:
- Multistage game has perfect information if at each stage, only one player has a non-trivial action set. ("alternating moves")
- We regard finite games although there is a generalization of backward induction to infinite games
- Observation: If games get long: "Rationality" of backward induction ("player moving at stage 0 must forecast that player at stage 1 will forecast that player at stage K-1 will forcast the move of player at stage K") becomes weaker with growing K.

Stage Games with Observed Actions

Backward Induction & Subgame Perfection

- Stackelberg EQ example: Alternating moves: backward induction easy:
- Multistage game has perfect information if at each stage, only one player has a non-trivial action set. ("alternating moves")
- We regard finite games although there is a generalization of backward induction to infinite games
- Observation: If games get long: "Rationality" of backward induction ("player moving at stage 0 must forecast that player at stage 1 will forecast that player at stage.....that player at stage K-1 will forcast the move of player at stage K") becomes weaker with growing K.

Backward Induction & Subgame Perfection

- Subgame perfection: Example: Two firms, two stages: First stage: firm 1 chooses to invest in technology or not; \cite{cond} Second stage: firm 1 and 2 choose output levels $\cite{q_1}$ and $\cite{q_2}$ simultaneously
- stage 2 simultaneous choice → Not perfect information
- But: Each history h¹ generates "normal" "strategic form-like" subgame → compute NE → subgame-perfect NE
- Stage k: History before k is h^k > Game from stage k on is G(h^k)





Backward Induction & Subgame Perfection

- Subgame perfection: Example: Two firms, two stages: First stage: firm 1 chooses to invest in technology or not; Second stage: firm 1 and 2 choose output levels q₁ and q₂ simultaneously
- stage 2 simultaneous choice → Not perfect information
- But: Each history h¹ generates "normal" "strategic form-like" subgame → compute NE → subgame-perfect NE
- Stage k: History before k is $h^k \rightarrow Game$ from stage k on is $G(h^k)$





- Subgame perfection: Example: Two firms, two stages: First stage: firm 1 chooses to invest in technology or not; Second stage: firm 1 and 2 choose output levels q₁ and q₂ simultaneously
- stage 2 simultaneous choice → Not perfect information
- But: Each history h¹ generates "normal" "strategic form-like" subgame → compute NE → subgame-perfect NE
- Stage k: History before k is $h^k \rightarrow Game$ from stage k on is $G(h^k)$



Backward Induction & Subgame Perfection

- Subgame perfection: Example: Two firms, two stages: First stage: firm 1 chooses to invest in technology or not; Second stage: firm 1 and 2 choose output levels q_1 and q_2 simultaneously
- stage 2 simultaneous choice → Not perfect information
- But: Each history h¹ generates "normal" "strategic form-like" subgame → compute NE → subgame-perfect NE



Stage Games with Observed Actions

Backward Induction & Subgame Perfection

- Subgame perfection: Example: Two firms, two stages: First stage: firm 1 chooses to invest in technology or not; Second stage: firm 1 and 2 choose output levels q_1 and q_2 simultaneously
- stage 2 simultaneous choice → Not perfect information
- But: Each history h¹ generates "normal" "strategic form-like" subgame → compute NE → subgame-perfect NE
- Stage k: History before k is $h^k \rightarrow Game$ from stage k on is $G(h^k)$

Multistage Games

- Stages (0,...,K); At each stage k: all players choose actions simultaneously; All players know actions from all players from all previous stages
- Action profile of stage k: $a^k = (a_1^k, a_2^k, ..., a_l^k)$
- History at end of stage k / beginning of stage k+1: $h^{k+1} = (a^0, a^1, ..., a^k); h^0 = \emptyset$
- Set of possible actions of player i if history hk: A;(hk)
- Possible: $A_i(h^k)=\{$ ", do nothing" $\} \rightarrow$ Previous example and generally games of ", alternating choice" included





- Subgame perfection: Example: Two firms, two stages: First stage: firm 1 chooses to invest in technology or not; Second stage: firm 1 and 2 choose output levels q_1 and q_2 simultaneously
- stage 2 simultaneous choice → Not perfect information
- But: Each history h¹ generates "normal" "strategic form-like" subgame → compute NE → subgame-perfect NE
- Stage k: History before k is $h^k \rightarrow Game$ from stage k on is $G(h^k)$

Backward Induction & Subgame Perfection

- Payoff in $G(h^k)$ is $u_i(h^{k+1})$ where $h^{k+1}=(h^k, a^{k+1}, a^{k+2}, ..., a^k)$
- Strategies in G(h^k): As before: maps from histories to actions, where histories are consistent with h^k
- → we can define NE in G(h^k) as before.
- Strategy profile s (whole game) induces strategy profile s $|h^k|$ on G(h^k): For all players $s_i|h^k$ is restriction of s_i to histories consistent with h^k

Backward Induction & Subgame Perfection

- Payoff in G(h^k) is $u_i(h^{K+1})$ where $h^{K+1}=(h^k, a^{k+1}, a^{k+2}, ..., a^k)$
- Strategies in G(h^k): As before: maps from histories to actions, where histories are consistent with h^k
- \rightarrow we can define NE in G(h^k) as before.
- Strategy profile s (whole game) induces strategy profile s $|h^k|$ on G(h^k): For all players $s_i|h^k$ is restriction of s_i to histories consistent with h^k



- Stage Games with Observed Actions

Backward Induction & Subgame Perfection

- Payoff in $G(h^k)$ is $u_i(h^{k+1})$ where $h^{k+1}=(h^k, a^{k+1}, a^{k+2}, ..., a^k)$
- Strategies in $G(h^k)$: As before: maps from histories to actions, where histories are consistent with h^k
- \rightarrow we can define NE in G(h^k) as before.
- Strategy profile s (whole game) induces strategy profile s $|h^k|$ on G(h^k): For all players $s_i|h^k$ is restriction of s_i to histories consistent with h^k





- Payoff in $G(h^k)$ is $u_i(h^{k+1})$ where $h^{k+1}=(h^k, a^{k+1}, a^{k+2}, ..., a^k)$
- Strategies in $G(h^k)$: As before: maps from histories to actions, where histories are consistent with h^k
- \rightarrow we can define NE in G(h^k) as before.
- Strategy profile s (whole game) induces strategy profile s $|h^k|$ on G(h^k): For all players $s_i|h^k$ is restriction of s_i to histories consistent with h^k



- Stage Games with Observed Actions

Backward Induction & Subgame Perfection

- Payoff in $G(h^k)$ is $u_i(h^{k+1})$ where $h^{k+1}=(h^k, a^{k+1}, a^{k+2}, ..., a^k)$
- Strategies in G(h^k): As before: maps from histories to actions, where histories are consistent with h^k
- \rightarrow we can define NE in G(h^k) as before.
- Strategy profile s (whole game) induces strategy profile s | h^k on G(h^k): For all players s_i | h^k is restriction of s_i to histories consistent with h^k



- Stage Games with Observed Actions

Backward Induction & Subgame Perfection

Strategy profile s (of whole multistage game with observed actions) is in subgame-perfect EQ if for all h^k the restriction $s \mid h^k$ to h^k on $G(h^k)$ is a NE of $G(h^k)$

- In games of perfect information ("alternating"): This definition reduces to backward induction
- Example from above: Two firms;; two stages;; cost per unit: 2;; stage 1: firm1 can invest in new technology: cost f which reduces cost per unit to 0;; linear demand: p(q)=14-q
 - Firm 1 does not invest: payoff is $[14-q_1-q_2]q_1 2q_1 = [12-q_1-q_2]q_1$
 - Firm 1 does invest: payoff is [14-q₁-q₂]q₁-f
 - Firm 2: playoff is [12-q₁-q₂]q₂





Backward Induction & Subgame Perfection

Strategy profile s (of whole multistage game with observed actions) is in subgame-perfect EQ if for all h^k the restriction $s \mid h^k$ to h^k on $G(h^k)$ is a NE of $G(h^k)$

- In games of perfect information ("alternating"): This definition reduces to backward induction
- Example from above: Two firms;; two stages;; cost per unit: 2;; stage 1: firm1 can invest in new technology: cost f which reduces cost per unit to 0;; linear demand: p(q)=14-q
 - Firm 1 does not invest: payoff is $[14-q_1-q_2]q_1 2q_1 = [12-q_1-q_2]q_1$
 - Firm 1 does invest: payoff is [14-q₁-q₂]q₁ -f
 - Firm 2: playoff is [12-q₁-q₂]q₂

Stage Games with Observed Actions

Backward Induction & Subgame Perfection

Strategy profile s (of whole multistage game with observed actions) is in subgame-perfect EQ if for all h^k the restriction $s \mid h^k$ to h^k on $G(h^k)$ is a NE of $G(h^k)$

- In games of perfect information ("alternating"): This definition reduces to backward induction
- Example from above: Two firms;; two stages;; cost per unit: 2;; stage 1: firm1 can invest in new technology: cost f which reduces cost per unit to 0;; linear demand: p(q)=14-q
 - Firm 1 does not invest: payoff is $[14-q_1-q_2]q_1 2q_1 = [12-q_1-q_2]q_1$
 - Firm 1 does invest: payoff is $[14-q_1-q_2]q_1$ -f
 - Firm 2: playoff is $[12-q_1-q_2]q_2$

Stage Games with Observed Actions

Backward Induction & Subgame Perfection

Strategy profile s (of whole multistage game with observed actions) is in subgame-perfect EQ if for all h^k the restriction $s \mid h^k$ to h^k on $G(h^k)$ is a NE of $G(h^k)$

- In games of perfect information ("alternating"): This definition reduces to backward induction
- Example from above: Two firms;; two stages;; cost per unit: 2;; stage 1: firm1 can invest in new technology: cost f which reduces cost per unit to 0;; linear demand: p(q)=14-q
 - Firm 1 does not invest: payoff is $[14-q_1-q_2]q_1 2q_1 = [12-q_1-q_2]q_1$
 - Firm 1 does invest: payoff is $[14-q_1-q_2]q_1$ -f
 - Firm 2: playoff is [12-q₁-q₂]q₂

- Stage Games with Observed Actions

Backward Induction & Subgame Perfection

- Firm 1 does not invest: reaction functions of firms are $r_i(q_{-i}) = 6 q_{-i}/2$ Cournot EQ: $(q_1^c, q_2^c) = (4;4)$: payoff is (16;16);
- Firm 1 does invest: reaction function of firm 1 changes to $r_1(q_2) = 7 q_2/2 \rightarrow NE$ changes to: (16/3; 10/3): firm 1's payoff is 256/9 f;
- \rightarrow Invest if 256/9 –f > 16
- Investing changes firm 1's payoff in two ways:
- (1) lower production cost
- (2) q_2 goes down because firm 1 can be more agressive ($r_1(q_2) > r_1(q_2)$ for all q_2)



Backward Induction & Subgame Perfection

Strategy profile s (of whole multistage game with observed actions) is in subgame-perfect EQ if for all h^k the restriction $s \mid h^k$ to h^k on $G(h^k)$ is a NE of $G(h^k)$

- In games of perfect information ("alternating"): This definition reduces to backward induction
- Example from above: Two firms;; two stages;; cost per unit: 2;; stage 1: firm1 can invest in new technology: cost f which reduces cost per unit to 0;; linear demand: p(q)=14-q
 - Firm 1 does not invest: payoff is $[14-q_1-q_2]q_1 2q_1 = [12-q_1-q_2]q_1$
 - Firm 1 does invest: payoff is [14-q₁-q₂]q₁ -f
 - Firm 2: playoff is [12-q₁-q₂]q₂

Stage Games with Observed Actions

- Firm 1 does not invest: reaction functions of firms are $r_i(q_{-i}) = 6 q_{-i}/2$ Cournot EQ: $(q_{-1}^c; q_{-2}^c) = (4;4)$: payoff is (16;16);
- Firm 1 does invest: reaction function of firm 1 changes to $r_1(q_2) = 7 q_2/2 \rightarrow NE$ changes to: (16/3; 10/3): firm 1's payoff is 256/9 - f;
- → Invest if 256/9 –f > 16
- Investing changes firm 1's payoff in two ways:
- (1) lower production cost
- (2) q_2 goes down because firm 1 can be more agressive $(r_1^{\sim}(q_2) > r_1(q_2))$ for all q_2)



