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Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

- Another example: Battle of the sexes
- Man & Woman; Ballet or Football

	B	F
F	0, 0	2, 1
B	1, 2	0, 0

- Another example: Game of chicken
- Driver 1 & Driver 2; Tough or Weak

	T	W
T	-1,-1	2, 1
W	1, 2	0, 0

Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

- Another example: Battle of the sexes
- Two pure NE: (F;F) and (B;B)
- One mixed NE: Indifference condition

→ Let $\sigma_1(F)=x$ and $\sigma_2(B)=y$ →
Player 1's indifference:
$0\,y + 2(1-y) = 1\,y + 0\,(1-y)$ → $y=2/3$
Player 2's indifference:
$0\,x + 2(1-x) = 1\,x + 0\,(1-x)$ → $x=2/3$
→ Mixed NE: ((2/3, 1/3); (2/3, 1/3))

	B	F
F	0, 0	2, 1
B	1, 2	0, 0

- Another example: Game of chicken
- (same reasoning) →

Mixed NE: ((1/2, 1/2); (1/2, 1/2))

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Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE

Risk Dominance / Pareto Optimality

- In this game: (Among others) two pure NE: (U,L) and (D,R); (U,L): Pareto dominates (D,R)
- But: For player 1 D is safer (guarrantees min payoff of 7) → If p(R) > 1/8 don‘t go for (U,L) → no certainty!
- Pregame-communication / agreement on (U,L) ?!

No: player 2 gains if player 1 plays U → player 2 will always tell „L“ regardless of true intentions
→ agreement is worthless

	L	R
U	9, 9	0, 8
D	8, 0	7, 7

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Games in Strategic Form & Nash Equilibrium

Nash Equilibrium: More than one NE
Risk Dominance / Pareto Optimality

	L	R
U	0,0,10	-5,-5,0
D	-5,-5,0	1,1,-5

A

	L	R
U	-2,-2,0	-5,-5,0
D	-5,-5,0	-1,-1,5

B

- Three player game: Two pure NE: (U,L,A) and (D,R,B); (and one mixed) ; (U,L,A) pareto-dominates (D,R,B)
- If player 3´s choice is fixed→ Two player game → (D,R) is pareto-dominant → if players 1 and 2 expect A : coordinate on (D,R).
- → concept of „coalition proof eq.“ (here (D,R,B))(see [1])

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Games in Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

		Player B q: L	Player B 1-q: R
Player A p	U	a_{UL}, b_{UL}	a_{UR}, b_{UR}
Player A 1-p	D	a_{DL}, b_{DL}	a_{DR}, b_{DR}

- Pure NE: One cell →
  For A: cell's payoff for A must be (weak) maximum over rows in that column
  For B: cell's payoff for B must be (weak) maximum over column in that row
- Example: (U,R) is pure NE if $a_{UR} \geq a_{DR}$ and $b_{UR} \geq b_{UL}$

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- Mixed NE: Start with player A's view
- If B plays mixed strategy (q, 1-q) or short q, A's best response is $(p^*(q)\ ,\ 1-p^*(q))$ or short $p^*(q)$.
- $u_A(p,q) = pq\ a_{UL} + p(1-q)\ a_{UR} + (1-p)q\ a_{DL} + (1-p)(1-q)\ a_{DR}$
  $= p\ [(a_{UL} - a_{UR} - a_{DL} + a_{DR})q + a_{UR} - a_{DR}\ ] + (a_{DL} - a_{DR})q + a_{DR}$
  $= p\ \delta(q) + (a_{DL} - a_{DR})q + a_{DR}$
- Now: case differentiation w.r.t. sign of $\delta(q)$: $u_A(p,q)$ will be maximal for:
  - $p^*=1$ if $\delta(q)$ is positive
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$\boxed{u_A(p,q) = p\ \delta(q) + (a_{DL} - a_{DR})q + a_{DR}}$

- Further case differentiation:
  - (1) If $\delta(q)=0$ for all q → for all q, A's best response are all p in [0,1]. (→ A is completely indifferent)
  - If δ is not identically zero: $\delta(q)$ is affine linear: Has at most one zero q^+ with $\delta(q^+)=0$. (exactly one if we set for $q^+ = \infty$ for $(a_{UL} - a_{UR} - a_{DL} + a_{DR}) = 0$) →
    - (2) If $q^+ \notin [0,1]$: A plays $p^*=1$ or $p^*=0$ depending on sign of $\delta(q)$. (→ A has a dominant pure strategy)
    - (3) If $q^+ \in [0,1]$: A plays $p^*=1$ or $p^*=0$ if $\delta(q) \neq 0$ (depending on sign of $\delta(q)$) and can play any p if $q=q^+$. (→ A plays strategically)

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    - (3) If $q^+ \in [0,1]$: A plays $p^*=1$ or $p^*=0$ if $\delta(q) \neq 0$ (depending on sign of $\delta(q)$) and can play any p if $q=q^+$. (→ A plays strategically)

Games in Strategic Form & Nash Equilibrium

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

$\boxed{u_A(p,q) = p\,\delta(q) + (a_{DL} - a_{DR})q + a_{DR}}$

- Further case differentiation:
  - (1) If $\delta(q)=0$ for all q → for all q, A's best response are all p in [0,1]. (→ A is completely indifferent)
  - If δ is not identically zero: $\delta(q)$ is affine linear: Has at most one zero q^+ with $\delta(q^+)=0$. (exactly one if we set for $q^+ = \infty$ for $(a_{UL} - a_{UR} - a_{DL} + a_{DR}) = 0$) →
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Vizualization of case (3)

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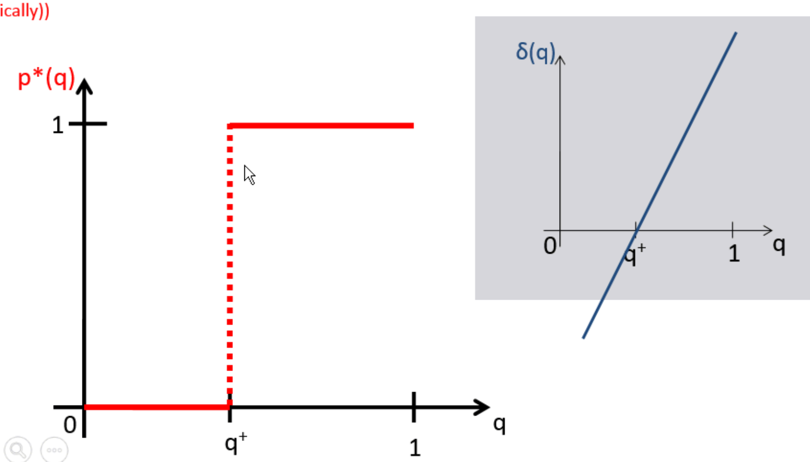

Mixed Nash Equilibrium: General Analysis for 2 x 2 Games (see [2])

Player B

Player A

		q	1-q
		L	R
p	U	a_{UL}, b_{UL}	a_{UR}, b_{UR}
1-p	D	a_{DL}, b_{DL}	a_{DR}, b_{DR}

- Now: Analysis for B:
- If A plays mixed strategy (p, 1-p) or short p, B's best response is $(q^*(p), 1-q^*(p))$ or short $q^*(p)$.
- $u_B(p,q) = q\,[(b_{UL} - b_{UR} - b_{DL} + b_{DR})p + b_{DL} - b_{DR}\,] + (b_{UR} - b_{DR})p + b_{DR}$
  $= q\,\gamma(p) + (b_{UR} - b_{DR})p + b_{DR}$
- Exactly the same case differentiations as in case of A
- Trivially: Nash-Equilibrium iff $q^*(p) = p^*(q)$.

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- Player 2's reaction on q_1 will be $s_2(q_1)$ → $u_i(q_1, q_2)=u_i(q_1, s_2(q_1))$
- Pure NE: Neither player can gain by switching from NE to other pure strategy:
  - (1) For given q_1: Maximize: $\max_{q_2} u_2(q_1, q_2)$ → we get Cournot reaction $r_2(q_1)=6 - q_1/2$; Now: Maximzie: $\max_{q_1} u_1(q_1, r_2(q_1))$ → $q^*_1 = 6$ → NE (Stackelberg EQ): $(q^*_1, r_2(q^*_1))$

Nash Equilibrium: Example: Cournot Competition

- Cournot model: Duopoly. Each of two firms (players) i produces same good.
- Output levels q_i are chosen from sets Q_i
- Cost of production is $c_i(q_i)$
- Market price is $p(q) = p(q_1+q_2)$
- Firm i's profit is then $u_i(q_1, q_2) = q_i\, p(q) - c_i(q_i)$
- Cournot reaction functions $r_1 : Q_2 \to Q_1$ and $r_2 : Q_1 \to Q_2$ specify optimal reaction on output level of opponent

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Inserting $r_2(q_1)$ for q_2

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gives the defining equation for $r_2(.)$.

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- Other NE: Cournot Equilibrium (from simultaneous play) $(q^c_1 ; q^c_2)=(4; 4)$;
- Why is this a NE? If player 2 plays q^c_2 (independent of q_1), player 1 must play $r_1(q^c_2) = q^c_1$ in order to maximize u_1. If player 1 plays q^c_1, player 2 can play any s_2 with $s_2(q^c_1) = q^c_2$ in order to maximize u_2, including s_2 = const = q^c_2.
- Stackelberg EQ $(q^*_1; r_2(q^*_1))$: payoff is (18;9);
  Cournot EQ $(q^c_1 ; q^c_2)$: payoff is (16;16);
- But: „player 2 plays q^c_2 (independent of q_1)" is empty threat: If player 1 plays $q_1 \neq q^c_1$, player 2 is better of with $r_2(q_1)$

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- Stages (0,...,K); At each stage k: all players choose actions simultaneously; All players know actions from all players from all previous stages
- Action profile of stage k: $a^k = (a^k_1, a^k_2, \ldots, a^k_I)$
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- Action profile of stage k: $a^k = (a^k_1, a^k_2, \ldots, a^k_I)$
- History at end of stage k / beginning of stage k+1: $h^{k+1} = (a^0, a^1, \ldots, a^k)$; $h^0 = \varnothing$
- Set of possible actions of player i if history h^k : $A_i(h^k)$
- Possible: $A_i(h^k)$={„do nothing"} → Previous example and generally games of „alternating choice" included

Multi-Stage Games with Observed Actions

Multistage Games

- Pure strategy for player i is contingent plan $s_i = (s^0_i, s^1_i, \ldots, s^K_i)$ where the s^k_i map the history to set of possible actions at each stage: $s^k_i(h^k) \in A_i(h^k)$;
- Mixed strategy $\sigma^k_i(h^k)$: define prob.-distribution over $A_i(h^k)$
- Thus: $a^k = s^k(h^k) = s^k(a^0, a^2, \ldots, a^{k-1})$;
- $(a^0, a^2, \ldots, a^j)$ is called path
- Overall Payoff of player i: $u_i(h^{K+1})$

Pure strategy NE: Strategy profile s so that for all players i:
$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ for all s'_i

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Multi-Stage Games with Observed Actions

Backward Induction & Subgame Perfection

- Stackelberg EQ example: Alternating moves: backward induction easy:
- Multistage game has perfect information if at each stage, only one player has a non-trivial action set. („alternating moves“)
- We regard finite games although there is a generalization of backward induction to infinite games
- Observation: If games get long: „Rationality“ of backward induction („player moving at stage 0 must forecast that player at stage 1 will forecast that player at stage......that player at stage K-1 will forcast the move of player at stage K“) becomes weaker with growing K.

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Backward Induction & Subgame Perfection

- Subgame perfection: Example: Two firms, two stages:
First stage: firm 1 chooses to invest in technology or not;
Second stage: firm 1 and 2 choose output levels q_1 and q_2 simultaneously
- stage 2 simultaneous choice → Not perfect information
- But: Each history h^1 generates „normal“ „strategic form-like“ subgame → compute NE → subgame-perfect NE
- Stage k: History before k is h^k → Game from stage k on is $G(h^k)$

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- Payoff in $G(h^k)$ is $u_i(h^{K+1})$ where $h^{K+1}=(h^k, a^{k+1}, a^{k+2}, \ldots, a^K)$
- Strategies in $G(h^k)$: As before: maps from histories to actions, where histories are consistent with h^k
- → we can define NE in $G(h^k)$ as before.
- Strategy profile s (whole game) induces strategy profile $s|h^k$ on $G(h^k)$: For all players $s_i|h^k$ is restriction of s_i to histories consistent with h^k

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Multi-Stage Games with Observed Actions

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> Strategy profile s (of whole multistage game with observed actions) is in subgame-perfect EQ if for all h^k the restriction $s|h^k$ to h^k on $G(h^k)$ is a NE of $G(h^k)$

- In games of perfect information („alternating"): This definition reduces to backward induction

- Example from above: Two firms;; two stages;; cost per unit: 2;; stage 1: firm1 can invest in new technology: cost f which reduces cost per unit to 0;; linear demand: $p(q)=14-q$
  - Firm 1 does not invest: payoff is $[14-q_1-q_2]q_1 - 2q_1 = [12-q_1-q_2]q_1$
  - Firm 1 does invest: payoff is $[14-q_1-q_2]q_1 - f$
  - Firm 2: playoff is $[12-q_1-q_2]q_2$

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- Firm 1 does not invest: reaction functions of firms are $r_i(q_{-i}) = 6 - q_{-i}/2$ → Cournot EQ: $(q^c_1; q^c_2)=(4;4)$: payoff is (16;16);
- Firm 1 does invest: reaction function of firm 1 changes to $r\sim_1(q_2) = 7 - q_2/2$ → NE changes to: (16/3; 10/3): firm 1's payoff is 256/9 - f;
- → Invest if $256/9 - f > 16$
- Investing changes firm 1's payoff in two ways:
  (1) lower production cost
  (2) q_2 goes down because firm 1 can be more agressive ($r\sim_1(q_2) > r_1(q_2)$ for all q_2)

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