


**Script** generated by TTT

Title: FDS (23.06.2017)

Date: Fri Jun 23 08:30:31 CEST 2017

Duration: 98:07 min

Pages: 53



8 Unbalanced BST

9 2-3 Trees

10 Red-Black Trees

**11 Tries and Patricia Tries**

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- 8 Unbalanced BST
- 9 2-3 Trees
- 10 Red-Black Trees
- 11 Tries and Patricia Tries**



Thys/Trie1  
Thys/Trie2



# Trie

Name: *reTRIEval*

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- Tries are search trees indexed by lists.

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Name: *reTRIEval*

- Tries are search trees indexed by lists.
- Each node maps the next element in the list to a subtrie.

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# Trie

Name: *reTRIEval*

- Tries are search trees indexed by lists.
- Each node maps the next element in the list to a subtrie.
- For simplicity we consider only binary tries:  
**datatype** *trie* = *Leaf* | *Node bool (trie × trie)*

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## Patricia trie

Sequences of edges without branching are compressed

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Sequences of edges without branching are compressed

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Sequences of edges without branching are compressed

**datatype** *ptrie* = *LeafP*  
| *NodeP* (*bool list*) *bool* (*ptrie* × *ptrie*)

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## Patricia trie

Sequences of edges without branching are compressed

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| *NodeP* (*bool list*) *bool* (*ptrie* × *ptrie*)

Name: PATRICIA = *Practical Algorithm To Retrieve Information Coded in Alphanumeric*

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# From trie to Patricia trie

and beyond

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## 11 Tries and Patricia Tries

Tries and Patricia Tries  
Data Refinement Basics  
Trie  
Patricia Trie  
Patricia Trie for Words

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## Data refinement

How to implement an abstract type  $A$  by a concrete (“refined”) type  $C$  with **invariant**  $inv_C :: C \Rightarrow bool$ :

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## Data refinement

How to implement an abstract type  $A$  by a concrete (“refined”) type  $C$  with **invariant**  $inv_C :: C \Rightarrow bool$ :

- Define an **abstraction function**  $abs_C :: C \Rightarrow A$
- Implement each interface function  $f_A$  on  $A$  by some  $f_C$  on  $C$

100



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- Prove that each  $f_C$  implements  $f_A$ :  
 $inv_C t \implies abs_C (f_C t) = f_A (abs_C t)$

100



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 $inv_C t \implies abs_C (f_C t) = f_A (abs_C t)$   
and preserves the invariant:  
 $inv_C t \implies inv_C (f_C t)$

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The  $f$ s may take more parameters

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 $inv_C t \implies abs_C (f_C t) = f_A (abs_C t)$   
and preserves the invariant:  
 $inv_C t \implies inv_C (f_C t)$

The  $f$ s may take more parameters  
and may return values not of type  $A/C$ .

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## Data refinement

Multiple refinement steps can help  
to reduce the complexity of correctness proofs

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## Trie

**datatype**  $trie = Leaf \mid Node\ bool\ (trie \times trie)$

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## Trie *isin*

$isin\ Leaf\ ks = False$

$isin\ (Node\ b\ (l,\ r))\ ks =$

(case  $ks$  of

$[] \Rightarrow b$

|  $k \# x \Rightarrow isin\ (if\ k\ then\ r\ else\ l)\ x$ )

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## Trie *insert*

$insert\ []\ Leaf = Node\ True\ (Leaf,\ Leaf)$

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## Trie insert

$insert \ [] \ Leaf = Node \ True \ (Leaf, \ Leaf)$   
 $insert \ [] \ (Node \ b \ lr) = Node \ True \ lr$

$insert \ (k \ \# \ ks) \ Leaf =$   
 $Node \ False$   
 (if  $k$  then  $(Leaf, \ insert \ ks \ Leaf)$   
 else  $(insert \ ks \ Leaf, \ Leaf)$ )

$insert \ (k \ \# \ ks) \ (Node \ b \ (l, \ r)) =$   
 $Node \ b \ (if \ k \ then \ (l, \ insert \ ks \ r) \ else \ (insert \ ks \ l, \ r))$

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## Correctness of *trie* w.r.t. *set*

$isin \ (insert \ as \ t) \ bs = (as = bs \vee isin \ t \ bs)$

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## Trie insert

$insert \ [] \ Leaf = Node \ True \ (Leaf, \ Leaf)$   
 $insert \ [] \ (Node \ b \ lr) = Node \ True \ lr$

$insert \ (k \ \# \ ks) \ Leaf =$   
 $Node \ False$   
 (if  $k$  then  $(Leaf, \ insert \ ks \ Leaf)$   
 else  $(insert \ ks \ Leaf, \ Leaf)$ )

$insert \ (k \ \# \ ks) \ (Node \ b \ (l, \ r)) =$   
 $Node \ b \ (if \ k \ then \ (l, \ insert \ ks \ r) \ else \ (insert \ ks \ l, \ r))$

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## 11 Tries and Patricia Tries

- Tries and Patricia Tries
- Data Refinement Basics
- Trie
- Patricia Trie
- Patricia Trie for Words

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## Patricia trie

```
datatype ptrie = LeafP  
  | NodeP (bool list) bool (ptrie × ptrie)
```

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## Splitting lists

```
split xs ys = (zs, xs', ys')
```

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## Splitting lists

```
split xs ys = (zs, xs', ys')  
iff zs is the longest common prefix of xs and ys
```

110



## Splitting lists

```
split xs ys = (zs, xs', ys')  
iff zs is the longest common prefix of xs and ys  
and xs'/ys' is the remainder of xs/ys
```

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## *insertP*

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## Splitting lists

$split\ xs\ ys = (zs, xs', ys')$   
 iff  $zs$  is the longest common prefix of  $x$ s and  $y$ s

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## Splitting lists

$split\ xs\ ys = (zs, xs', ys')$   
 iff  $zs$  is the longest common prefix of  $x$ s and  $y$ s  
 and  $xs'/ys'$  is the remainder of  $x$ s/ $y$ s

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## $abs\_ptrie :: ptrie \Rightarrow trie$

$abs\_ptrie\ LeafP = Leaf$   
 $abs\_ptrie\ (NodeP\ ps\ b\ (l, r)) =$   
 $prefix\_trie\ ps\ (Node\ b\ (abs\_ptrie\ l, abs\_ptrie\ r))$

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## Correctness of *ptrie* w.r.t. *trie*

$$isinP\ t\ ks = isin\ (abs\_ptrie\ t)\ ks$$

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## Correctness of *ptrie* w.r.t. *trie*

$$isinP\ t\ ks = isin\ (abs\_ptrie\ t)\ ks$$
$$abs\_ptrie\ (insertP\ ks\ t) = insert\ ks\ (abs\_ptrie\ t)$$

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### ① Tries and Patricia Tries

- Tries and Patricia Tries
- Data Refinement Basics
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## Patricia trie for words

```
datatype ptrieW = LeafW
  | NodeW (32 word) (32 word) bool
  (ptrieW × ptrieW)
```

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## Patricia trie for words

```
datatype ptrieW = LeafW
  | NodeW (32 word) (32 word) bool
  (ptrieW × ptrieW)
```

```
to_bl :: 'a word ⇒ bool list
```

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## *isinW*

```
isinW LeafW k = False
```

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## *isinW*

```
isinW LeafW k = False
```

```
isinW (NodeW p m b (l, r)) k =
  (if b then k = p
   else if match_prefix p m k
     then isinW (if go_left m k then l else r) k
     else False)
```

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## Correctness of *ptrie* w.r.t. *trie*

```
inv_ptrieW t ⇒
isinW t k = isinL (abs_ptrieW t) (rev (to_bl k))
```

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## Correctness of *ptrie* w.r.t. *trie*

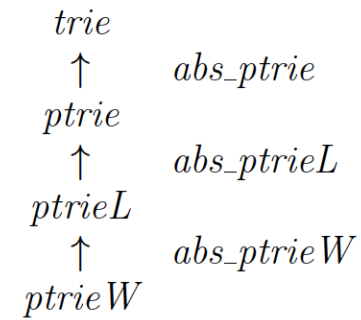
$$\begin{aligned} \text{inv\_ptrie } W \ t &\implies \\ \text{isin } W \ t \ k &= \text{isin } L \ (\text{abs\_ptrie } W \ t) \ (\text{rev } (\text{to\_bl } k)) \end{aligned}$$

Proof uses intermediate version of Patricia trie for lists where nodes contain complete prefixes (*isinL*).

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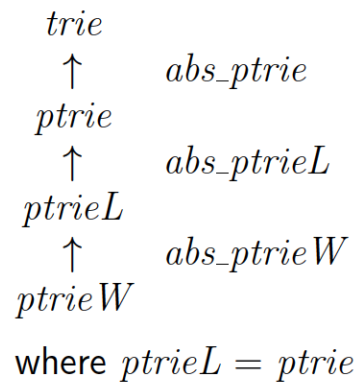
## Refinement hierarchy



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## Refinement hierarchy



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$$\text{inv\_ptrie } W :: \text{ptrie } W \implies \text{bool}$$

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## Refinement hierarchy

$$\begin{array}{c}
 \text{trie} \\
 \uparrow \text{ abs\_ptrie} \\
 \text{ptrie} \\
 \uparrow \text{ abs\_ptrieL} \\
 \text{ptrieL} \\
 \uparrow \text{ abs\_ptrieW} \\
 \text{ptrieW}
 \end{array}$$

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$$\text{inv\_ptrie } W :: \text{ ptrie } W \Rightarrow \text{ bool}$$

$$\text{inv\_ptrie } W \text{ Leaf } W = \text{ True}$$

$$\begin{aligned}
 \text{inv\_ptrie } W (\text{Node } W \ p \ m \ b \ (l, r)) = \\
 (\text{inv\_pref } p \ m \wedge \text{inv\_mask } m \wedge \text{inv\_ptrie } W \ l \wedge \\
 \text{inv\_ptrie } W \ r)
 \end{aligned}$$

$$\begin{aligned}
 \text{inv\_pref } p \ m = \\
 (\text{let } ps = \text{to\_bl } p; \ ms = \text{to\_bl } m; \\
 \quad n = \text{length } (\text{dropWhile } \text{Not } ms); \\
 \quad ps' = \text{take } (n + 1) \ ps \\
 \text{in set } ps' = \{\text{False}\})
 \end{aligned}$$

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$$\text{inv\_ptrie } W :: \text{ ptrie } W \Rightarrow \text{ bool}$$

$$\text{inv\_ptrie } W \text{ Leaf } W = \text{ True}$$

$$\begin{aligned}
 \text{inv\_ptrie } W (\text{Node } W \ p \ m \ b \ (l, r)) = \\
 (\text{inv\_pref } p \ m \wedge \text{inv\_mask } m \wedge \text{inv\_ptrie } W \ l \wedge \\
 \text{inv\_ptrie } W \ r)
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 \text{inv\_pref } p \ m = \\
 (\text{let } ps = \text{to\_bl } p; \ ms = \text{to\_bl } m; \\
 \quad n = \text{length } (\text{dropWhile } \text{Not } ms); \\
 \quad ps' = \text{take } (n + 1) \ ps \\
 \text{in set } ps' = \{\text{False}\})
 \end{aligned}$$

$$\text{inv\_mask } m = ([b \leftarrow \text{to\_bl } m . b] = [\text{True}])$$

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$$\text{abs\_ptrie } W :: \text{ ptrie } W \Rightarrow \text{ ptrie}$$

$$\text{abs\_ptrie } W \text{ Leaf } W = \text{ Leaf } P$$

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## Patricia trie for words

```
datatype ptrieW = LeafW  
| NodeW (32 word) (32 word) bool  
  (ptrieW × ptrieW)
```

```
to_bl :: 'a word ⇒ bool list
```



## Further space optimization?

```
datatype ptrieW2 =  
  LeafW2 |  
  SingW2 (32 word) |  
  NodeW2 (32 word) (32 word) (ptrieW2 × ptrieW2)
```